Primitive Constructive Logic

Distinction, Report, Set, Index, Number, Operation, Constructive Zero, and the Foundations of Mathematical Meaning

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Complete proof v1.0

A constructive proof of critical-half confinement for indexed completed-zeta zero reports.

Abstract

This paper gives a dependency-ordered construction of Primitive Constructive Logic. It begins from realized distinction rather than from a pre-existing number line, coordinate grid, field, or set-theoretic universe. A report is intelligible only when its components are distinguishable; distinguishability requires whole report placements; and those placements supply a pre-coordinate geometry prior to Cartesian representation. Objects, sets, indices, counts, arithmetic operations, ratios, completion, algebraic carriers, analysis, and zero are then reconstructed in that order.

The central invariant is that a mathematical report is not exhausted by its projected value. A value projection may identify two occurrences, but occurrence identity also retains report position, source, type, construction route, residual state, boundary, and declared closure. Thus extensional collapse is a later quotient, not a primitive identity rule. Addition is derived as a source-preserving join, multiplication as indexed equal repetition, subtraction as retained removal, and division as complete group extraction with quotient and remainder. Fractions are retained ratio relations rather than partial objects; reciprocals are compiled unit-to-denominator ratios; and the scalar half is derived from complementary balance through the integer condition $2A - B = 0$.

Zero is treated with the same discipline. Literal absence has no report, index, or membership. Any written zero or empty-set mark is a distinguishable typed report object representing absence, cancellation, remainder closure, lack of admitted members, or complete operator closure according to its index. Consequently there is a unique zero value only after a chosen value projection, while there may be many distinct indexed zero occurrences before that projection.

The final application constructs finite zeta source ledgers, their convergence and analytic continuation, the completed zeta report, reflection, conjugation, source-lane transport, and indexed zero equality. For a finite horizontal coordinate $s = (A + iT)/B$, reflection sends the native imbalance $H_B = 2A - B$ to $-H_B$. The completed reflection ledger has a fixed-sign radial imbalance whenever $H_B \neq 0$. Equality with the indexed complete zero occurrence therefore forces $H_B = 0$, whose normalized coordinate report is $\Re(s) = 1/2$. The theorem proved is the indexed PCL critical-half theorem for every nontrivial completed-zeta zero report.

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Part I
The existence of mathematical report

1 Primitive foundation

Axiom 1.1 — Realized distinction

At least one distinction is realized.

This axiom does not initially assert a number, coordinate, set, duration, or spatial axis. It asserts only that some distinguishable state exists.

Definition 1.1 — Existence

An **existence** is anything capable of participating in a realized distinction.

An existence need not be material. It may be a thing, state, measurement, idea, relation, or pattern.

Definition 1.2 — Thinker

A **thinker** is an existence capable of distinguishing.

Definition 1.3 — Thought

A **thought** is a distinction realized by a thinker.

The statement

I think, therefore I am

is therefore reconstructed as:

$$\text{a realized distinction occurs} \Longrightarrow \text{an existence capable of distinction occurs.}$$

The cogito establishes existence through distinction. It does not yet establish number, arithmetic, geometry, ordering, or comparison.

Definition 1.4 — Observation

An **observation** is a thought rendered as a distinction in reality.

An observation therefore contains at least:

- that which is distinguished;
- that from which it is distinguished;
- and the distinction relating them.

Observation is already relational.

Definition 1.5 — Measurement

A **measurement** is an observation that distinguishes an object, state, placement, pattern, or relation.

Location is a measurement because a location distinguishes one placement from another.

A measurement need not initially be numerical. Numerical measurement is a later report of a previously distinguished relation.

Definition 1.6 — Joined measurement

A **joined measurement** is a relation in which two or more distinguishers retain an equivalent observation.

This does not require that the distinguishers possess identical thoughts. It requires that their separately rendered observations admit a shared distinguishable relation.

Definition 1.7 — Report

A **report** is a distinguishable rendering of one or more thoughts or measurements.

A report must be understandable. Understandability requires its components to remain distinguishable.

Therefore:

no report exists without distinguishable report placements.

Every report is consequently geometric in the primitive sense: its distinguishable components occupy distinct placements within the report.

This geometry is prior to Cartesian coordinates. It requires neither x- nor y-axes.

2 Object and set

Definition 2.1 — Object

An **object** is any existence admitted as one distinguishable component of a report.

An object may be:

- a physical thing;
- an observation;
- a measurement;
- an idea;
- a relation;
- a state;
- or a pattern.

Calling something an object means that it has been distinguished as one reportable existence.

Definition 2.2 — Report index

Every distinguishable component of a report occupies one complete report position.

That position is its **index** within that rendering.

For a report R, write:

$$R = (r_i)_{i \in I_R},$$

where:

- I_R is the family of whole report positions;
- r_i is the object rendered at position i.

There is no fractional admission of an object into a report. An object is either rendered as one distinguishable component or it is not rendered.

A purported "half-index" must either remain indistinguishable, in which case it is not a report position, or become distinguishable, in which case it is another whole position in a refined report.

Definition 2.3 — Primitive set

A **primitive set** is an understandable report grouping distinguishable objects.

Formally, a primitive set-report is:

$$S = (I_S, \sigma_S),$$

where:

$$\sigma_S : I_S \to \mathcal{O}$$

assigns one distinguishable object to each whole report index.

Because a set does not contain duplicate members under the same identity criterion, σ_S is injective:

$$\sigma_S(i) = \sigma_S(j) \Longrightarrow i = j.$$

Thus a set is not an unrendered metaphysical collection. It is an understandable grouping of distinctly placed objects.

Definition 2.4 — Reindexing

Two set-reports may contain the same objects in different rendered orders:

$$(a, b, c), (c, a, b).$$

A conventional unordered-set report identifies these under reindexing.

Let

$$S \sim_{\text{perm}} T$$

when a bijection between their index carriers preserves the reported objects.

The conventional unordered set is then the projection:

$$[S]_{\text{perm}}.$$

This projection discards the order of rendering. It does not prove that no indexed rendering occurred.

Therefore:

unordered means reindexing-invariant, not unindexed.

3 Absence, empty reports, and zero

Definition 3.1 — Literal absence

Literal absence is the absence of any distinguishable object or report.

Literal absence cannot render itself.

It has:

- no report placement;
- no index;
- no identity within a set;
- and no symbol of its own.

The moment a mark is used to denote absence, that mark is a distinguishable report object.

Theorem 3.1 — Unreportability of a literal empty set

A literally empty set cannot be rendered as an understandable report.

Proof.

A report requires at least one distinguishable rendered component. A literally empty set contains no component capable of occupying a report position. Therefore it cannot itself produce an understandable rendering. □

Definition 3.2 — Empty-set token

The symbol

$$\varnothing$$

is not literal absence. It is one distinguishable report object whose interpreted meaning is:
no member-object has been admitted to the referenced grouping.
Thus:

$$\{\varnothing\}$$

contains one report object: the absence token.

The token represents absence, but is not identical to literal absence.

Definition 3.3 — Constructive zero

A **constructive zero** is a distinguishable report object representing the absence of a declared contribution, remainder, member, change, or surviving obligation.

If

$$0_S \in S,$$

then 0_S is one whole indexed object in the report S.
It may report:

$$\text{no contribution},$$

but it cannot be literal nothingness because it has been distinguished and admitted.
Therefore:

$$\boxed{0_S \in S \Longrightarrow 0_S \text{ is constructive.}}$$

This is the foundational theorem that eventually reaches RH.

4 First geometric relations

Definition 4.1 — Point

A **point** is one distinguishable indexed placement in a report.

If r_i is rendered at report position i, then the placement relation may be denoted:

$$p_i = (i, r_i).$$

A point is not initially a dimensionless Cartesian coordinate. It is the minimum distinguishable placement event.

Definition 4.2 — Successor placement

Because report components are rendered distinguishably, a report may retain a successor relation:

$$p_i \to p_{i+1}.$$

One successor transition is the minimum complete positional difference:

$$\Delta i = 1.$$

Definition 4.3 — Primitive line

A **line** is a measurement relating the placement of one distinguishable point to another.

A primitive line is the complete relation between directly adjacent report placements:

$$\ell_i = (p_i, p_{i+1}).$$

Its index difference is one:

$$(i+1) - i = 1.$$

There is no half of an adjacency event. Refining an adjacency introduces additional whole placements and a new ratio between the old and refined reports.

Part II
Distinguishable Multiplicity and the First Mathematical Constructions

5 Distinguishable occurrences

A report cannot contain an object without rendering that object at a distinguishable report position. Therefore an object's reported value and its occurrence within the report must be kept separate.

Definition 5.1 — Report occurrence

Let R be an understandable report. An **occurrence** in R is a pair

$$\omega_i = (i, v_i),$$

where:

- i is the occurrence's report index;
- v_i is the value, identity, pattern, or type reported at that index.

Two occurrences are distinct whenever their indices are distinct:

$$i \neq j \Longrightarrow \omega_i \neq_{\mathrm{C}} \omega_j.$$

This remains true even when their reported values agree:

$$v_i = v_j.$$

Accordingly, value equality does not imply occurrence identity:

$$v_i = v_j \backslash \text{centernot} \Longrightarrow \omega_i = \omega_j.$$

The distinction is between

$$\omega_i =_{report} \omega_j$$

and

$$\omega_i =_{\mathrm{C}} \omega_j.$$

The first means that the two occurrences produce the same value or type report. The second means that they are the same indexed occurrence.

5.1 Repetition requires distinct indices

Consider the primitive set-report

$$R = [a_0, a_1],$$

where

$$\pi(a_0) = \pi(a_1) = a.$$

The two occurrences have the same projected report a, but they occupy different positions:

$$ind(a_0) = 0, ind(a_1) = 1.$$

Therefore,

$$a_0 \neq_{\mathrm{C}} a_1.$$

The report contains two objects:

$$\mid R \mid_C = 2.$$

By contrast,

$$R^{'} = [a_0]$$

contains one occurrence:

$$\mid R^{'} \mid_C = 1.$$

Hence,

$$[a_0, a_1] \neq_{\mathrm{C}} [a_0].$$

If both occurrences are projected only to their common value a, an extensional report may collapse them:

$$\pi_{ext}([a_0, a_1]) = \{a\} = \pi_{ext}([a_0]).$$

That equality belongs to the projection. It is not equality of the original reports.

Theorem 5.1 — Occurrence preservation

Let

$$R = (I_R, v_R)$$

be a report with index carrier I_R and value-reporting map

$$v_R : I_R \to V.$$

The constructive multiplicity of R is

$$N_C(R) =\mid I_R \mid .$$

Its projected extensional value collection is

$$E(R) = im(v_R).$$

Then

$$\mid E(R) \mid \leq N_C(R),$$

with strict inequality whenever distinct indices carry the same projected value.

Proof.

Suppose

$$i \neq j$$

but

$$v_R(i) = v_R(j).$$

The constructive report retains both i and j, so they contribute two occurrences to $N_C(R)$. The image $E(R)$, however, contains only one copy of their common projected value. Therefore projection may reduce multiplicity. □

Corollary 5.1

A value projection cannot be used to count occurrences unless it is injective:

$$v_R(i) = v_R(j) \Longrightarrow i = j.$$

If injectivity has not been proved, extensional value count and constructive occurrence count are different quantities.

6 Primitive set-reports

Definition 6.1 — Primitive set-report

A **primitive set-report** is an understandable grouping of one or more indexed object occurrences:

$$S = (I_S, v_S).$$

Each index in I_S is occupied by one distinguishable occurrence.

A nonempty primitive set-report therefore satisfies

$$\mid I_S \mid \geq 1.$$

A set-report is geometric because its members occupy distinguishable placements within the report.

Definition 6.2 — Constructive set equality

Two primitive set-reports

$$S = (I_S, v_S), T = (I_T, v_T)$$

are constructively equal only when there exists an index-preserving identification between their occurrences that preserves all declared relevant structure.

At minimum, constructive equality requires:

$$\mid I_S \mid = \mid I_T \mid$$

and a bijection

$$\phi : I_S \to I_T$$

such that

$$v_S(i) = v_T(\phi(i))$$

for every $i \in I_S$.

If source, order, orientation, or construction history is declared relevant, those structures must also be preserved.

Definition 6.3 — Reindexing equality

Two reports may be treated as equivalent under reindexing if a bijection preserves the reported occurrences while changing only their presentation order.

For example,

$$[a_0, a_1, a_2]$$

and

$$[a_2, a_0, a_1]$$

may be reindexing-equivalent:

$$[a_0, a_1, a_2] \sim_{\text{perm}} [a_2, a_0, a_1].$$

This does not imply that the original report order never existed. It means only that the chosen later equality relation ignores that order.

Thus:

unordered means invariant under reindexing, not devoid of indices.

7 Literal absence and the impossibility of a reported empty set

Definition 7.1 — Literal absence

Literal absence is the absence of every distinguishable occurrence and every report.

Literal absence has:

- no index;
- no placement;
- no identity;
- no membership;
- and no reportable mark.

It cannot appear as an object in an understandable statement.

Proposition 7.1 — Every denotation of emptiness is nonempty as a report

Any mark used to denote emptiness is itself a distinguishable occurrence.

For example, the mark

$$\varnothing$$

occupies a report position. Therefore the mark is not literal absence. It is a report object interpreted as indicating that no member occurrences have been admitted under a referenced condition.

Similarly, the symbol

$$0$$

is not literal absence. It is a distinguishable report object interpreted as indicating no contribution of a declared type.

Consequently,

$$\{0\}$$

contains one report object:

$$| \{0\} |_C = 1,$$

even though the contribution reported by that object may be

$$\mu(0) = 0.$$

These are different statements:

$$\boxed{\text{one object representing no contribution}}$$

and

$$\boxed{\text{no object and no report.}}$$

Theorem 7.1 — Constructive zero necessity

If a zero appears in a set-report,

$$0_S \in S,$$

then 0_S is a whole indexed occurrence in S. It is therefore a constructive report object and cannot be literal absence.

8 Iteration and the first numbers

Number cannot be introduced before distinguishable multiplicity.

Definition 8.1 — Single occurrence

Let

$$A = [a_0].$$

The report A contains one occurrence.

The primitive numerical report associated with this completed occurrence is denoted

$$1.$$

Thus 1 does not create the occurrence. It reports that exactly one distinguishable occurrence has been retained.

Definition 8.2 — Iteration

Given a report A, an iteration of A is a new occurrence produced under the same declared type or construction rule while receiving a new index.

If

$$A = [a_0],$$

then one additional iteration produces

$$Iter(A) = [a_0, a_1],$$

where

$$a_0 \neq_{\mathrm{C}} a_1$$

but perhaps

$$\pi(a_0) = \pi(a_1).$$

The count report is then

$$2.$$

Accordingly,

$$2$$

means that two distinguishable occurrences have been completed and retained.

Definition 8.3 — Constructive natural number

A constructive natural number n is the report of a set containing n whole distinguishable occurrences:

$$n \equiv \mid I_S \mid .$$

The numeral is a representation of the multiplicity. It is not the source of that multiplicity.

Thus the primitive order is

$$\text{occurrence} \rightarrow \text{iteration} \rightarrow \text{count} \rightarrow \text{numeral}.$$

9 Joining reports and primitive addition

Definition 9.1 — Source-preserving join

Let

$$A = (I_A, v_A), B = (I_B, v_B).$$

Their source-preserving join is

$$A \sqcup B = (\{A\} \times I_A) \cup (\{B\} \times I_B).$$

The source labels ensure that occurrences from A and B remain distinguishable even when they carry equal projected values.

For example, let

$$A = [a_0], B = [a_1].$$

Even if

$$\pi(a_0) = \pi(a_1),$$

their join contains two occurrences:

$$A \sqcup B = [a_0, a_1].$$

Therefore,

$$\mid A \sqcup B \mid = \mid A \mid + \mid B \mid .$$

Definition 9.2 — Primitive addition

Addition is the count report produced by source-preserving joining:

$$\mid A \sqcup B \mid = \mid A \mid + \mid B \mid .$$

For two singleton reports,

$$\mid A \mid = 1, \mid B \mid = 1,$$

and therefore

$$\mid A \sqcup B \mid = 2.$$

The equation

$$1 + 1 = 2$$

is a compressed report of the prior construction

$$[a_0] \sqcup [a_1] = [a_0, a_1].$$

Addition does not create the objects. It reports the multiplicity of their retained join.

Theorem 9.1 — Addition preserves constructive multiplicity

For disjointly indexed reports A and B,

$$N_C(A \sqcup B) = N_C(A) + N_C(B).$$

Projecting equal-valued occurrences to a single extensional value before counting destroys this law.

For example,

$$[a_0] \sqcup [a_1]$$

contains two occurrences, while the projected value set may be only

$$\{a\}.$$

The extensional projection therefore cannot replace the source-preserving join when multiplicity matters.

10 Removal and primitive subtraction

Definition 10.1 — Constructive removal

Let A be a report and let B identify occurrences already contained in A.

Constructive subtraction removes the occurrences indexed by B from A, while retaining the relation describing what was removed and from where.

Write:

$$A \ominus B = R.$$

The complete subtraction state is

$$(A, B, R).$$

The remainder R contains the occurrences of A not removed through B.

Example.

Let

$$A = [a_0, a_1, a_2],$$

and remove

$$B = [a_1].$$

Then

$$A \ominus B = [a_0, a_2].$$

The count report is

$$3 - 1 = 2.$$

The subtraction report is meaningful because the removed occurrence existed previously in A.

Exact removal

If every occurrence of A is removed by an occurrence-identical copy of A, then no positive contribution remains:

$$A \ominus A = 0_A.$$

Here 0_A is the constructive zero report indexed to the removal of A. It certifies complete removal of the declared source occurrences.

It is not literal nonexistence, because the subtraction relation remains understandable:

$$(A, A, 0_A).$$

11 Two distinguishable placements and the primitive line

Every nonempty report contains indexed placements. Two distinguishable placements admit a relation.

Definition 11.1 — Primitive point

A **primitive point** is one indexed occurrence considered as a distinguishable placement within a report.

Let

$$p_i = (i, v_i).$$

The point does not require Cartesian coordinates. Its index already distinguishes its placement from other report placements.

Definition 11.2 — Primitive line

Given two distinguishable points

$$p_i \neq p_j,$$

a primitive line is the measurement relation connecting their placements:

$$\ell_{ij} = Rel(p_i, p_j).$$

If the two points are directly adjacent in the declared report order, their index difference is one transition:

$$\mid ind(p_j) - ind(p_i) \mid = 1.$$

This is the minimum complete line relation.

Proposition 11.1

Two distinguishable points can construct a line but cannot construct an area.

With two points,

$$P = \{p_0, p_1\},$$

there is only one undirected pairwise relation:

$$\ell_{01}.$$

No closed cycle exists.

Therefore:

$$2 \text{ points} \longrightarrow 1 \text{ line},$$

but not an enclosed area.

12 Three distinguishable placements and minimum area

Definition 12.1 — Closed three-point relation

Let

$$P = \{p_0, p_1, p_2\}.$$

Define the boundary relations

$$\ell_{01}, \ell_{12}, \ell_{20}.$$

These form the cycle

$$p_0 \to p_1 \to p_2 \to p_0.$$

This is the minimum finite line relation capable of distinguishing an interior from an exterior.

Theorem 12.1 — Minimum area closure

A polygonal area requires at least three distinguishable placements and at least three boundary relations.

Proof.

One point provides no relation.

Two points provide one line relation but leave the boundary open.

Three points permit three pairwise boundary relations arranged as a closed cycle. Therefore three is the least occurrence count capable of polygonal enclosure. □

Equal-type construction

The three points need not possess different projected types.

Let

$$\pi(p_0) = \pi(p_1) = \pi(p_2) = a.$$

They remain distinct because

$$0 \neq 1 \neq 2.$$

Thus repeated type identity does not prevent area construction. Area arises from distinguishable placement and closed incidence, not from variation in reported labels.

Three-singleton formulation

Let

$$A = [a_0], B = [a_1], C = [a_2].$$

Suppose they are equivalent under the selected type or size report:

$$A =_\pi B =_\pi C,$$

but remain source-distinct:

$$A \neq_{\mathrm{C}} B, B \neq_{\mathrm{C}} C, C \neq_{\mathrm{C}} A.$$

Their join is

$$A \sqcup B \sqcup C = [a_0, a_1, a_2].$$

Adding the three pairwise boundary relations constructs the minimum area closure.
One-plus-two formulation
Equivalently, let

$$A = [a_0], B = [a_1, a_2].$$

Then

$$| A | = 1, | B | = 2,$$

and

$$A \sqcup B = [a_0, a_1, a_2].$$

The same minimum three-point closure can then be formed.

13 Equal-sized set-reports and the grid

Definition 13.1 — Placement product
Let

$$X = (I_X, v_X), Y = (I_Y, v_Y).$$

Their placement product is the report of every ordered cross-relation:

$$X \times Y = \{(x_i, y_j) : i \in I_X,\ j \in I_Y\}.$$

Each ordered pair occupies one whole product index.
Example 13.1
Let

$$X = [0, 1], Y = [0, 1].$$

Then

$$X \times Y = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The product contains four occurrences:

$$| X \times Y | = 4.$$

The mixed ordered pairs remain distinct:

$$(0, 1) \neq_{\mathrm{C}} (1, 0).$$

Even if a later commutative scalar map reports

$$0 + 1 = 1 + 0,$$

the two source routes are not constructively identical.

Definition 13.2 — Grid

A **grid** is a placement product of two indexed reports.

An equal grid satisfies

$$\mid I_X \mid = \mid I_Y \mid .$$

A square grid is therefore a product construction over two equal-sized index carriers. It is not primitive merely because its sides are reported as equal.

Addition report on the 2×2grid

Define

$$\pi_+(i, j) = i + j.$$

Then:

$$\pi_+(0, 0) = 0, \pi_+(0, 1) = 1, \pi_+(1, 0) = 1, \pi_+(1, 1) = 2.$$

The scalar report is

$$[0, 1, 1, 2].$$

The scalar value 1 has two distinct preimages:

$$\pi_+^{-1}(1) = \{(0, 1), (1, 0)\}.$$

Therefore the scalar report 1 does not uniquely identify its constructive route.

This is the elementary form of the projection-fiber distinction later used throughout Primitive Constructive Algebra.

14 First projection-collapse law

Theorem 14.1 — Equal reports do not imply equal constructions

Let

$$\pi : \mathcal{C} \to V$$

be any report map from constructive objects to projected values.

If

$$\pi(C_1) = \pi(C_2),$$

it does not follow that

$$C_1 = C_2.$$

A proof of constructive equality requires a further theorem showing that π is injective over the relevant domain.

The 2×2 grid gives the first explicit example:

$$\pi_+(0,1) = 1 = \pi_+(1,0),$$

while

$$(0,1) \neq_{\mathrm{C}} (1,0).$$

Thus commutative scalar equality is a report equality, not source identity.
This elementary distinction will later control:

- multiplication;
- fractions;
- quotient–remainder states;
- square-grid balance;
- complex conjugation;
- reflected zeta terms;
- and the critical-half report.

Part III
Repetition, Product, Division, and Retained Remainder

15 Repeated equal joining

Addition joins already distinguishable occurrences. Multiplication requires a further construction: the repeated joining of equal-sized occurrence families.

Definition 15.1 — Constructive copy

Let

$$A = (I_A, v_A)$$

be a primitive set-report.
A **constructive copy** of A is a new report

$$A^{(k)} = (I_A^{(k)}, v_A^{(k)})$$

together with an occurrence-preserving correspondence

$$c_k : I_A \to I_A^{(k)}.$$

For each $i \in I_A$,

$$v_A(i) =_\pi v_A^{(k)}(c_k(i)),$$

while the source-indexed occurrences remain distinct:

$$(A, i) \neq_{\mathrm{C}} (A^{(k)}, c_k(i)).$$

Thus a copy repeats the declared structure or type of A; it does not reuse the same occurrence as though one object occupied multiple indices simultaneously.

Definition 15.2 — Repetition index

Let

$$K = [k_0, k_1, \ldots, k_{n-1}]$$

be a report containing n distinguishable repetition indices.
For each $k_j \in K$, construct one indexed copy $A^{(k_j)}$.
The repeated equal join of A over K is

$$Rep_K(A) = \bigsqcup_{k \in K} A^{(k)}.$$

Every occurrence in the result retains both:

- its position inside A;
- the repetition index selecting its copy.

Its complete source address is therefore

$$(k, i) \in K \times I_A.$$

Theorem 15.1 — Repetition count

If A contains m occurrences and K contains n repetition indices, then

$$Rep_K(A)$$

contains one occurrence for every ordered source pair

$$(k, i) \in K \times I_A.$$

Hence its constructive count is reported as

$$n \cdot m.$$

Proof.

Each of the n repetition indices selects one complete copy of the m-occurrence report A. The copies are source-disjoint because their repetition indices differ. Thus there is one retained occurrence for each pair (k, i), giving nm occurrences. □

The scalar product

$$nm$$

is therefore a count report of a prior indexed repetition construction.
It is not the construction itself.

16 Primitive multiplication

Definition 16.1 — Multiplication by repetition

Let m report the count of a primitive occurrence family A, and let n report the count of a repetition-index family K.

The multiplication report

$$n \cdot m$$

is the count of the repeated equal join

$$\mathrm{Rep}_K(A).$$

Thus multiplication is derived from:

$$\begin{aligned}\text{distinguishable occurrence} &\longrightarrow \text{equal constructive copy} \longrightarrow \text{indexed repetition}\\ &\longrightarrow \text{source-preserving join} \longrightarrow \text{count report.}\end{aligned}$$

Multiplication cannot be primitive before these constructions have been established.

Example 16.1

Let

$$A = [a_0, a_1]$$

and

$$K = [k_0, k_1, k_2].$$

Then

$$Rep_K(A)$$

contains the six source-addressed occurrences

$$(k_0, a_0), (k_0, a_1),(k_1, a_0), (k_1, a_1),(k_2, a_0), (k_2, a_1).$$

Its scalar count report is

$$3 \cdot 2 = 6.$$

The number 6 does not retain which of the six source pairs produced it. It is a projection of the six-cell repetition ledger.

17 Multiplication as a placement product

Repeated joining and the grid product are two views of the same indexed construction.

Let

$$K = [k_0, \ldots, k_{n-1}]$$

and

$$A = [a_0, \ldots, a_{m-1}].$$

The complete multiplication ledger is

$$K \times A = \{(k_i, a_j)\}.$$

The repeated-join view groups the cells by repetition index:

$$K \times A = \bigsqcup_{k \in K} (\{k\} \times A).$$

The product view retains the entire cross-relation at once.
Therefore:

$$\boxed{\text{multiplication ledger} = \text{ordered placement product},}$$

while

$$\boxed{\text{scalar multiplication} = \text{count projection of that product.}}$$

17.1 Multiplication is not primitive area

The grid used to display multiplication is not the primitive source of area.
Multiplication needs only:

- an indexed repetition family;
- an indexed repeated family;
- and their ordered pairings.

Area additionally requires closed boundary relations.
Thus:

$$K \times A$$

is a multiplication ledger whether or not its cells are interpreted as a geometric enclosure.
A scalar product does not become area merely because it can be arranged on a rectangular grid.

18 Orientation and commutative projection

Let

$$A \times B = \{(a_i, b_j)\}$$

and

$$B \times A = \{(b_j, a_i)\}.$$

These are not literally the same source ledger. Their ordered components have been reversed.
Definition 18.1 — Transposition
Define the transposition map

$$\tau_{A,B} : A \times B \to B \times A$$

by

$$\tau_{A,B}(a_i, b_j) = (b_j, a_i).$$

This map is a bijection.

Therefore the two products have the same count:

$$| A \times B | = | B \times A | .$$

The scalar report is consequently commutative:

$$| A | \cdot | B | = | B | \cdot | A | .$$

But the source ledgers remain orientation-distinct:

$$(a_i, b_j) \neq_{\mathrm{C}} (b_j, a_i)$$

unless a separate equivalence deliberately identifies them.

Theorem 18.1 — Constructive commutativity

Multiplication is commutative at the scalar-count level because transposition pairs every cell of $A \times B$ with exactly one cell of $B \times A$.

Therefore:

$$A \times B \cong B \times A,$$

but generally

$$A \times B \neq_{\mathrm{C}} B \times A.$$

The symbol $\cong$ reports structure-preserving correspondence. It should not be silently replaced by full source identity.

Example 18.1

Let

$$A = B = [0, 1].$$

The cells

$$(0, 1)$$

and

$$(1, 0)$$

are exchanged by transposition. They remain separately indexed product occurrences.

A later scalar map may report them equally, but their product orientations have not thereby ceased to exist.

19 One-copy and zero-copy reports

19.1 Multiplication by one

Let K_1 contain one repetition index:

$$K_1 = [k_0].$$

Then

$$K_1 \times A = \{(k_0, a) : a \in A\}.$$

There is an occurrence-preserving correspondence

$$(k_0, a) \longleftrightarrow a.$$

Therefore the count report satisfies

$$1\cdot \mid A \mid = \mid A \mid .$$

The one-copy product may be canonically normalized back to A, but the full ledger may still retain the fact that a one-copy operation occurred.

19.2 Multiplication by zero

A zero repetition report means that no copy-selection event has occurred.

It cannot be represented by a literally empty understandable report. Instead, the complete multiplication state contains a constructive closure marker

$$0_{Rep(A)}$$

reporting:

No occurrence of A was selected into the repeated join.

Thus:

$$0\cdot \mid A \mid = 0$$

is the scalar projection of a typed no-copy state.

The zero result is not literal nothingness. It is a distinguishable report of zero completed repetitions of the declared object family.

20 Associative regrouping

Consider three indexed reports A, B, and C.

The product

$$(A \times B) \times C$$

contains occurrences of the form

$$((a, b), c).$$

The product

$$A \times (B \times C)$$

contains occurrences of the form

$$(a, (b, c)).$$

These are differently nested source reports.
Define the reassociation map

$$\alpha((a, b), c) = (a, (b, c)).$$

This map is bijective, so the two products have the same count.
Therefore:

$$(\mid A \mid \cdot \mid B \mid)\cdot \mid C \mid = \mid A \mid \cdot(\mid B \mid \cdot \mid C \mid).$$

Again, scalar associativity follows from a structure-preserving correspondence:

$$(A \times B) \times C \cong A \times (B \times C),$$

not from literal identity of their nested source syntax.

21 Constructive distribution

Let $B \sqcup C$ be a source-preserving join.
Then

$$A \times (B \sqcup C)$$

contains two source-distinct classes:

$$\{(a, b) : a \in A, b \in B\}$$

and

$$\{(a, c) : a \in A, c \in C\}.$$

Therefore:

$$A \times (B \sqcup C) \cong (A \times B) \sqcup (A \times C).$$

Taking counts gives:

$$\mid A \mid (\mid B \mid + \mid C \mid) = \mid A \mid\mid B \mid + \mid A \mid\mid C \mid .$$

Distributivity is thus a decomposition theorem for a source-preserving product ledger.
It is not merely a symbolic rearrangement.

Part IV
Comparison, Extraction, Division, and Remainder

22 Primitive comparison by complete containment

Division requires comparison, but comparison cannot yet rely on fractions, decimal magnitudes, or a continuous number line.

Definition 22.1 — Constructive containment comparison

Let A and B be finite occurrence reports.

Write

$$A \preceq_C B$$

when every occurrence of A can be matched injectively to a distinct occurrence of B.

This means that B contains enough occurrence positions to receive one complete copy of A.

Write

$$A \prec_C B$$

when

$$A \preceq_C B$$

but no occurrence-preserving bijection exists between them.

The scalar reports are:

$$| A | \leq | B |$$

and

$$| A | < | B | .$$

Thus numerical order is derived from whole-occurrence containability.

Definition 22.2 — Equal size

Two reports have equal size when each can be injected into the other, equivalently when there exists an occurrence-preserving bijection between their index carriers.

This is size equality, not necessarily full constructive identity.

23 Complete group extraction

Let X be a dividend occurrence report and B a positive divisor template.

A division construction asks:

How many complete source-disjoint copies of B can be identified inside X, and which occurrences remain unassigned?

Definition 23.1 — Extraction copy

An extraction copy of B from X is an injective occurrence map

$$e_k : I_B \to I_X$$

whose image identifies one complete B-sized group inside X.

Distinct extraction copies must have disjoint images:

$$im(e_j) \cap im(e_k) = \text{no shared positive occurrence}$$

for $j \neq k$.

Definition 23.2 — Extraction family

Let

$$Q = [q_0, q_1, \ldots, q_{r-1}]$$

index a family of complete, source-disjoint extractions of B from X.

The extracted occurrence family is represented by

$$Q \times B.$$

The occurrences of X not assigned to any extraction form the positive remainder carrier R.

The complete decomposition is

$$X \cong (Q \times B) \sqcup R.$$

This is the constructive source of the scalar equation

$$\mid X \mid = \mid Q \mid\mid B \mid + \mid R \mid .$$

24 Primitive division

Definition 24.1 — Constructive quotient

The **constructive quotient** is the indexed report Q counting how many complete copies of B were extracted from X.

The scalar quotient is

$$q = \mid Q \mid .$$

The quotient is not initially a fractional scalar. It is a count of completed whole-group extractions.

Definition 24.2 — Constructive remainder

The **constructive remainder** is the source-retained report of dividend occurrences not assigned to a complete extracted copy of B.

Write

$$R = R_{X,B,Q,\gamma},$$

where γ records the declared extraction or grouping rule.

The complete division state is therefore

$$\mathcal{D}(X, B; \gamma) = (X, B, Q, R, \gamma).$$

Its scalar projection is

$$\mid X \mid = q \mid B \mid + \mid R \mid .$$

Remainder condition
Extraction is complete when the remainder cannot contain another full divisor copy:

$$B \not\preceq_C R.$$

At the scalar-count level, this is reported as

$$0 \leq \mid R \mid < \mid B \mid .$$

The inequality is not primitive. It summarizes the constructive fact that no further complete B-sized extraction is available.

25 Quotient uniqueness and decomposition nonuniqueness

A critical distinction must be retained.

For fixed finite counts

$$\mid X \mid = A, \mid B \mid = B > 0,$$

the quotient count q and remainder count r satisfying

$$A = qB + r, 0 \leq r < B$$

are unique.

But the source-level grouping need not be unique.

Example 25.1
Let

$$X = [x_0, x_1, x_2, x_3]$$

and let the divisor size be two.

One extraction rule may group:

$$[x_0, x_1], [x_2, x_3].$$

Another may group:

$$[x_0, x_2], [x_1, x_3].$$

Both produce the same scalar quotient:

$$q = 2$$

and the same scalar remainder:

$$r = 0.$$

But the grouping ledgers differ.

Therefore the scalar division state does not uniquely recover the constructive partition unless a grouping rule γ has been declared.

Theorem 25.1 — Scalar uniqueness does not imply ledger uniqueness

The quotient and remainder counts are uniquely determined by the dividend and divisor counts.

The source decomposition is uniquely determined only when the admissible extraction rule is itself deterministic or all admissible decompositions are proved equivalent for the property under study.

Thus:

$$(A, B) \longmapsto (q, r)$$

may be unique while

$$(X, B) \longmapsto (Q, R, \gamma)$$

retains multiple possible construction histories.

26 Exact division and constructive zero

Suppose every dividend occurrence is assigned to a complete divisor copy.

Then there is no positive remainder occurrence.

The division state must nevertheless report that the extraction closed exactly.

Definition 26.1 — Exact-remainder zero

The exact-remainder zero is the typed closure object

$$0_{X:B,\gamma}$$

whose meaning is:

Under extraction rule γ, every occurrence of X has been assigned to one complete copy of B, and no positive occurrence remains unassigned.

The complete division state is

$$\mathcal{D}(X, B; \gamma) = (X, B, Q, 0_{X:B,\gamma}, \gamma).$$

Its scalar projection is

$$A = qB + 0.$$

The zero is not an absent report. It is the report certifying exact closure of a declared extraction obligation.

Theorem 26.1 — Exact division requires an indexed zero

An exact quotient cannot be reported without distinguishing the condition that no positive remainder survives.

Therefore exact division necessarily produces a constructive zero typed to:

- the dividend;
- the divisor;
- the quotient family;
- and the extraction rule.

Two exact divisions may both project to scalar remainder 0 while remaining different zero-closure events:

$$0_{X:B,\gamma} \neq_{\mathrm{C}} 0_{Y:C,\delta}.$$

Their scalar zero reports agree, but their source obligations differ.

27 Division is not multiplication by a primitive reciprocal

At this stage, no reciprocal object has been constructed.

Therefore division cannot be primitively defined by

$$A/B = A \cdot B^{-1}.$$

That formula presupposes:

- a reciprocal carrier;
- multiplication on that carrier;
- an identity object;
- and a proof that the reciprocal closes the declared multiplication obligation.

Primitive division instead begins with:

$$\begin{aligned}\text{whole dividend occurrences} &\longrightarrow \text{complete divisor-copy extraction}\\ &\longrightarrow \text{quotient count} \longrightarrow \text{retained remainder.}\end{aligned}$$

Only later may a reciprocal be introduced as a report that reproduces the quotient behavior inside a declared extended carrier.

Thus:

$$\boxed{\text{division precedes reciprocal notation constructively.}}$$

A reciprocal is a compiled relation derived from division; it cannot be used to define division without circularity.

28 The complete quotient–remainder identity

The complete constructive division law is:

$$\boxed{X \cong (Q \times B) \sqcup R}$$

with

$$B \not\preceq_C R.$$

Its scalar-count projection is:

$$\boxed{A = qB + r, 0 \leq r < B.}$$

The first expression retains:

- the dividend source X;
- the divisor template B;
- the quotient index family Q;
- every extracted product cell;
- the remainder source R;
- and the extraction rule γ.

The second expression retains only their counts.
Therefore:

$$\boxed{A = qB + r}$$

is not the full division object. It is the scalar report of a prior source-preserving decomposition.

Part V
Ratios, Fractions, Reciprocals, and the Constructive Half

29 Ratio as a retained two-report relation

Division produces more information than a scalar quotient. For finite reports X and B, the complete division state retains

$$X \cong (Q \times B) \sqcup R,$$

with no further complete copy of B constructible from R.
When

$$| X | < | B |,$$

no complete divisor copy can be extracted. The quotient count is therefore zero, but the dividend has not disappeared:

$$Q = 0_{X:B}, R = X.$$

The scalar quotient alone reports

$$0.$$

The complete state reports

$$| X | = 0 \cdot | B | + | X | .$$

This distinction is the primitive source of a proper ratio.
Definition 29.1 — Constructive ratio

Let A and B be finite occurrence reports, with B carrying positive count:

$$\mid B \mid > 0.$$

The **constructive ratio of** A**to** B is the retained relation

$$\mathcal{R}(A, B) = (A, B, \mathcal{D}(A, B)),$$

where $\mathcal{D}(A, B)$ is the quotient–remainder decomposition of A by B.
The ratio retains:

- the numerator source A;
- the denominator source B;
- the complete quotient family Q;
- the remainder R;
- and the grouping rule used in the division.

A ratio is therefore not initially a freestanding magnitude. It is a relation between two declared constructions.

Definition 29.2 — Ratio notation

The notation

$$\frac{\mid A \mid}{\mid B \mid}$$

is the scalar report of the constructive ratio

$$\mathcal{R}(A, B).$$

The fraction bar does not perform an unexplained operation. It abbreviates the prior division relation.

Thus the correct dependency order is

$$(A, B) \to \mathcal{D}(A, B) \to \mathcal{R}(A, B) \to \frac{\mid A \mid}{\mid B \mid}.$$

The fraction notation is last.

30 Proper ratios retain a full residual carrier

Suppose A contains fewer occurrences than B:

$$0 <\mid A \mid <\mid B \mid .$$

No complete B-sized group can be extracted from A. Therefore

$$q = 0, R = A.$$

The complete finite state is

$$| A |= 0\cdot | B | + | A | .$$

Example 30.1 — The ratio one to two
Let

$$U = [u_0]$$

be a singleton occurrence report, and let

$$D = [d_0, d_1]$$

be a two-occurrence denominator report.
Then

$$| U |= 1, | D |= 2.$$

Dividing U by D permits no complete extraction of D:

$$U \cong (0_{U:D} \times D) \sqcup U.$$

Its scalar quotient–remainder state is

$$1 = 0 \cdot 2 + 1.$$

The exact constructive ratio is therefore represented by

$$(q, R; B) = (0, 1; 2).$$

Only after the denominator is retained does the coordinate notation become

$$\frac{1}{2}.$$

The pair

$$(q, R) = (0, 1)$$

is insufficient by itself, because it would also arise from

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5},$$

and every other ratio in which one occurrence is smaller than the denominator family.
Their complete states are different:

$$(0, 1; 2), (0, 1; 3), (0, 1; 4), (0, 1; 5).$$

Therefore:

$$\boxed{\text{the denominator is part of the ratio object.}}$$

The fraction is not merely a quotient and remainder. It is a typed relation among dividend, divisor, quotient, and residual carrier.

31 A fraction does not divide an occurrence

The notation

$$\frac{1}{2}$$

is often spoken of as though one primitive object had been cut into two primitive half-objects. That interpretation is not required by the constructive ratio.

In the state

$$(0, 1; 2),$$

the residual occurrence remains whole:

$$R = [u_0].$$

Nothing has become half an occurrence. Instead, the whole singleton is being compared with a whole two-occurrence denominator family.

Thus:

$$\boxed{\frac{1}{2} \text{ reports one whole occurrence relative to a two-occurrence carrier.}}$$

It does not report:

$$\boxed{\text{one occurrence whose index has been divided into two.}}$$

A fractional report changes the comparison carrier. It does not create fractional occurrence identity.

Proposition 31.1 — Whole-index preservation

Let

$$\mathcal{R}(A, B)$$

be a constructive ratio. Every occurrence in A and B retains a whole index in its source report.

Therefore ratio formation does not produce partial indices.

Proof.

Ratio formation performs comparison and grouping between already indexed reports. It introduces no operation that partially admits an occurrence. Every member of A and B remains distinguishable at a whole source position. The fractional scalar is a report of their relative counts, not a subdivision of their indices. □

32 Ratio equality

Two ratio reports may have equal scalar values while retaining different source constructions.

Let

$$\mathcal{R}(A, B)$$

and

$$\mathcal{R}(C, D)$$

be ratios with positive denominator counts.

Definition 32.1 — Scalar ratio equality

The ratios are **scalar-equal** when their cross-products have equal constructive count:

$$\mid A \times D \mid = \mid C \times B \mid .$$

Equivalently,

$$\mid A \parallel D \mid = \mid C \parallel B \mid .$$

Write

$$\mathcal{R}(A, B) =_{ratio} \mathcal{R}(C, D).$$

This equality is established without first assuming a field of fractions.

Example 32.1

Let

$$\mid A \mid = 1, \mid B \mid = 2,$$

and

$$\mid C \mid = 2, \mid D \mid = 4.$$

Then

$$\mid A \times D \mid = 1 \cdot 4 = 4,$$

while

$$\mid C \times B \mid = 2 \cdot 2 = 4.$$

Therefore:

$$\mathcal{R}(A, B) =_{ratio} \mathcal{R}(C, D).$$

Their coordinate reports are

$$\frac{1}{2} = \frac{2}{4}.$$

However, the underlying source ledgers need not be constructively identical:

$$(A, B) \neq_{\mathrm{C}} (C, D).$$

One ratio compares a singleton with a two-occurrence carrier. The other compares a two-occurrence report with a four-occurrence carrier.

Thus:

equal fractions need not be equal constructions.

They belong to the same scalar ratio class because their cross-product counts balance.

33 Ratio equivalence by common replication

The most direct source of equivalent fractions is common replication.

Let K be a positive repetition-index report. Construct

$$K \times A$$

and

$$K \times B.$$

Their counts are

$$| K || A |, | K || B | .$$

The ratio relation is unchanged at the scalar level:

$$\mathcal{R}(A, B) =_{ratio} \mathcal{R}(K \times A, K \times B).$$

Theorem 33.1 — Common-replication invariance

For every positive repetition carrier K,

$$\frac{| A |}{| B |} = \frac{| K || A |}{| K || B |}.$$

Proof.

The cross-products are

$$A \times (K \times B)$$

and

$$(K \times A) \times B.$$

By reassociation and transposition, both contain one occurrence for each triple

$$(k, a, b) \in K \times A \times B.$$

They therefore have equal constructive count. □

The theorem explains fraction scaling constructively. Multiplying numerator and denominator by the same positive number means replicating both source reports by the same indexed carrier.

The scalar value remains invariant because the same replication structure is added to both sides of the comparison.

34 Common cancellation

The reverse operation is possible only when both numerator and denominator have actually been constructed through a shared repetition carrier.

Suppose

$$A \cong K \times A',$$

and

$$B \cong K \times B^{'}.$$

Then the common carrier K may be removed from the scalar ratio report:

$$\mathcal{R}(A, B) =_{ratio} \mathcal{R}(A^{'}, B^{'}).$$

Definition 34.1 — Constructive cancellation

Constructive cancellation removes a common declared replication carrier from both sides of a ratio.

It is valid when the decompositions

$$A \cong K \times A^{'}, B \cong K \times B^{'}$$

have been established.

It is not justified merely because two scalar symbols appear to possess a common written factor.

Thus cancellation is a theorem about retained product structure.

Definition 34.2 — Reduced ratio

A ratio

$$\mathcal{R}(A, B)$$

is **constructively reduced** relative to the declared grammar when no non-singleton positive report K has been shown to satisfy

$$A \cong K \times A^{'}, B \cong K \times B^{'}.$$

This definition does not yet require a greatest-common-divisor algorithm. It requires only that no common replication carrier remain in the proven construction.

35 Mixed quotient–ratio reports

Suppose the dividend contains one or more complete divisor copies together with a nonzero remainder:

$$A = qB + R, 0 <\mid R \mid <\mid B \mid .$$

The complete report may be separated into:

- q complete copies of B;
- one retained proper ratio $\mathcal{R}(R, B)$.

Thus:

$$\mathcal{R}(A, B) = q + \mathcal{R}(R, B).$$

Its coordinate report is

$$\frac{A}{B} = q + \frac{R}{B}.$$

This is not a decomposition of A into abstract scalar pieces. It is a report of the actual source decomposition

$$A \cong (Q \times B) \sqcup R.$$

Example 35.1
Let

$$| A |= 5, | B |= 2.$$

Then two complete copies of B can be extracted, leaving one occurrence:

$$5 = 2 \cdot 2 + 1.$$

The full ratio state is

$$\mathcal{R}(5, 2) = 2 + \mathcal{R}(1, 2).$$

Its coordinate notation is

$$\frac{5}{2} = 2 + \frac{1}{2}.$$

The final $\frac{1}{2}$ remains the relation

$$(0, 1; 2).$$

No half-occurrence has been introduced.

36 Reciprocal reports

A reciprocal cannot be primitive because it presupposes the ratio relation from which it is derived.
Definition 36.1 — Unit report
Let U be a singleton report:

$$| U |= 1.$$

The unit report supplies the comparison source for reciprocal construction.
Definition 36.2 — Reciprocal of a positive report
Let B be a positive occurrence report:

$$| B |> 0.$$

The **reciprocal report of** B is the constructive ratio

$$Rec(B) = \mathcal{R}(U, B).$$

Its coordinate notation is

$$\frac{1}{| B |}.$$

Thus the reciprocal of B is not an independently existing scalar created by inversion. It is the relation of one unit occurrence to the positive denominator carrier B.

For a two-occurrence carrier:

$$Rec(B_2) = \mathcal{R}(U, B_2),$$

whose quotient–remainder state is

$$(0, 1; 2).$$

The notation is

$$\frac{1}{2}.$$

Proposition 36.1

The reciprocal report retains its denominator source.
Therefore:

$$Rec(B_2) \neq_{\mathrm{C}} Rec(B_3),$$

even though both produce quotient zero and remainder one before the divisor is restored.

37 Multiplication by a reciprocal

Once reciprocal reports exist, the conventional identity

$$A/B = A \cdot B^{-1}$$

can be recovered as a later theorem.
Let

$$Rec(B) = \mathcal{R}(U, B).$$

Multiplying A by this ratio produces

$$\mathcal{R}(A \times U, B).$$

Because U is a singleton,

$$A \times U \cong A.$$

Therefore:

$$A \cdot Rec(B) =_{ratio} \mathcal{R}(A, B).$$

Its scalar report is:

$$\mid A \mid \cdot \frac{1}{\mid B \mid} = \frac{\mid A \mid}{\mid B \mid}.$$

Theorem 37.1 — Reciprocal multiplication is compiled division

Multiplication by a reciprocal reproduces the scalar ratio only after:

1. a positive denominator report B exists;
2. division has been defined;

3. the unit-to-B ratio has been constructed;
4. ratio multiplication has been defined;
5. the singleton product has been normalized.

Thus:

$$\boxed{A/B = A \cdot B^{-1}}$$

is a compiled identity. It is not the primitive definition of division.

38 Zero ratio and the absence of a reciprocal

Let 0_A be a constructive zero reporting no numerator contribution relative to denominator B.

The ratio

$$\mathcal{R}(0_A, B)$$

has scalar report

$$\frac{0}{\mid B \mid} = 0.$$

This is well formed whenever B is positive.

The source meaning is:

No numerator occurrence contributes relative to the declared positive denominator carrier.

38.1 Why zero has no reciprocal

To construct the reciprocal of a numerical zero, one would require

$$\mathcal{R}(U, 0).$$

But numerical zero supplies no positive divisor carrier. There is no complete occurrence template against which extraction can be performed.

The zero symbol remains an understandable object, but its reported numerical contribution is zero. The symbol's existence does not turn zero contribution into a positive divisor family.

Therefore the positive-divisor compiler cannot be invoked.

Theorem 38.1 — Division by zero is not a ratio state

A constructive ratio

$$\mathcal{R}(A, B)$$

requires

$$\mid B \mid > 0.$$

If the denominator contribution is zero, no positive grouping template exists, no maximal complete extraction family is defined, and no remainder bound of the form

$$0 \leq R < B$$

can be constructed.
Therefore:

$$\boxed{\frac{A}{0} \text{ is not a completed constructive ratio.}}$$

It is not automatically infinity, zero, or an arbitrary value. It is a failure to supply the positive divisor structure required by the division operation.

Corollary 38.1

Zero has no multiplicative reciprocal:

$$0^{-1}$$

is undefined in the ratio grammar.

This is not because the zero symbol is literally absent. It is because its declared numerical contribution cannot serve as a positive divisor carrier.

39 Ratio multiplication

Let

$$\mathcal{R}(A, B)$$

and

$$\mathcal{R}(C, D)$$

have positive denominator reports B and D.
Their product ratio is constructed by pairing numerator sources and denominator sources:

$$\mathcal{R}(A, B) \otimes \mathcal{R}(C, D) = \mathcal{R}(A \times C, B \times D).$$

The scalar report is

$$\frac{\mid A \mid}{\mid B \mid} \cdot \frac{\mid C \mid}{\mid D \mid} = \frac{\mid A \mid\mid C \mid}{\mid B \mid\mid D \mid}.$$

The product retains four source families:

- A;
- B;
- C;
- D.

A reduced scalar fraction may later cancel common replication carriers, but that reduction cannot erase the original multiplication history unless the relevant property factors through the reduced report.

40 Ratio addition

Ratio addition requires a shared comparison carrier.

Let

$$\mathcal{R}(A, B)$$

and

$$\mathcal{R}(C, D)$$

be constructive ratios.

Construct the common denominator carrier

$$B \times D.$$

Replicate A across D:

$$A \times D,$$

and replicate C across B:

$$C \times B.$$

The numerator contributions may then be joined:

$$(A \times D) \sqcup (C \times B).$$

Definition 40.1 — Constructive ratio sum

Define

$$\mathcal{R}(A, B) \oplus \mathcal{R}(C, D) = \mathcal{R}((A \times D) \sqcup (C \times B), B \times D).$$

Its scalar projection is

$$\frac{|\, A \,|}{|\, B \,|} + \frac{|\, C \,|}{|\, D \,|} = \frac{|\, A \,||\, D \,| + |\, C \,||\, B \,|}{|\, B \,||\, D \,|}.$$

This familiar formula is therefore derived from:

1. common carrier construction;
2. numerator replication;
3. source-preserving join;
4. ratio reporting.

The two numerator lanes remain distinguishable:

$$A \times D$$

and

$$C \times B.$$

Their scalar sum does not prove that they arose through the same route.

41 Ratio subtraction

Suppose a constructive matching removes the contribution of

$$C \times B$$

from

$$A \times D.$$

Then the ratio difference is

$$\mathcal{R}(A, B) \ominus \mathcal{R}(C, D) = \mathcal{R}((A \times D) \ominus (C \times B), B \times D).$$

The scalar report is

$$\frac{\mid A \mid}{\mid B \mid} - \frac{\mid C \mid}{\mid D \mid} = \frac{\mid A \parallel D \mid - \mid C \parallel B \mid}{\mid B \parallel D \mid}.$$

If the numerator lanes close exactly, the result is a typed constructive zero:

$$0_{AD:CB}.$$

The zero records which two ratio contributions were matched.

Thus two fraction differences may both project to scalar zero while retaining distinct source cancellation histories.

Part VI
Complement, Balance, and the Constructive Half

42 Complement inside a finite carrier

Let B be a positive finite report, and let A be identified as a source-preserving subreport of B.

The complement of A in B is the remainder report C satisfying

$$B \cong A \sqcup C.$$

At the scalar level:

$$\mid B \mid = \mid A \mid + \mid C \mid .$$

The ratio of A to B is

$$\mathcal{R}(A, B),$$

and the ratio of its complement is

$$\mathcal{R}(C, B).$$

Their sum is one complete carrier:

$$\mathcal{R}(A,B) \oplus \mathcal{R}(C,B) = \mathcal{R}(B,B).$$

The ratio $\mathcal{R}(B,B)$ reports one complete extraction:

$$\mid B \mid = 1 \cdot \mid B \mid + 0.$$

Its coordinate report is

$$1.$$

Therefore:

$$\frac{\mid A \mid}{\mid B \mid} + \frac{\mid C \mid}{\mid B \mid} = 1.$$

The unit arises from complete carrier closure.

43 Constructive balance

Definition 43.1 — Balanced complement partition
A decomposition

$$B \cong A \sqcup C$$

is **balanced** when A and C have equal size:

$$\mid A \mid = \mid C \mid .$$

Because

$$\mid B \mid = \mid A \mid + \mid C \mid,$$

balance gives

$$\mid B \mid = \mid A \mid + \mid A \mid .$$

After multiplication has been constructed, this may be reported as

$$\mid B \mid = 2 \mid A \mid .$$

The native balance condition is therefore

$$\mid A \mid - \mid C \mid = 0,$$

or equivalently,

$$2 \mid A \mid - \mid B \mid = 0.$$

Definition 43.2 — Integer balance lane
Define the integer balance report

$$H(A;B) = 2 \mid A \mid - \mid B \mid .$$

Equivalently, if C is the complement of A in B,

$$H(A;B) =\mid A \mid - \mid C \mid .$$

Then:

$$H(A;B) = 0$$

if and only if the two complementary occurrence families have equal size.
This balance statement contains no primitive fraction.

44 Derivation of the half

Assume

$$B \cong A \sqcup C$$

and

$$\mid A \mid = \mid C \mid .$$

Then:

$$\mid B \mid = 2 \mid A \mid .$$

The ratio relation of A to B satisfies

$$2 \mid A \mid = \mid B \mid .$$

Its coordinate report is therefore

$$\frac{\mid A \mid}{\mid B \mid} = \frac{1}{2}.$$

Theorem 44.1 — Constructive half theorem
A ratio reports one half precisely when its numerator carrier is size-equal to its retained complement inside the denominator carrier.
Equivalently:

$$\boxed{\frac{\mid A \mid}{\mid B \mid} = \frac{1}{2} \iff B \cong A \sqcup C \text{ with } \mid A \mid = \mid C \mid .}$$

The fraction $\frac{1}{2}$ is therefore the normalized scalar report of the integer balance condition

$$2 \mid A \mid - \mid B \mid = 0.$$

It is not a primitive object and not half of a report index.
Proof.
If $B \cong A \sqcup C$ and $\mid A \mid = \mid C \mid$, then

$$\mid B \mid = \mid A \mid + \mid A \mid = 2 \mid A \mid .$$

Cross-multiplying the ratios

$$\mathcal{R}(A, B)$$

and

$$\mathcal{R}(U, D_2),$$

where U is a singleton and D_2 is a two-occurrence carrier, gives

$$\mid A \mid \cdot 2 = \mid B \mid \cdot 1.$$

Therefore the ratios are scalar-equal.
Conversely, if

$$2 \mid A \mid = \mid B \mid$$

and A is retained as a subreport of B, then the complement C has count

$$\mid C \mid = \mid B \mid - \mid A \mid = 2 \mid A \mid - \mid A \mid = \mid A \mid .$$

Thus the two complementary carriers balance. □

45 The half does not identify either side

A balanced decomposition retains two source-distinct complementary families:

$$A \neq_{\mathrm{C}} C,$$

even though

$$\mid A \mid = \mid C \mid .$$

Their ratios have equal scalar reports:

$$\frac{\mid A \mid}{\mid B \mid} = \frac{\mid C \mid}{\mid B \mid} = \frac{1}{2}.$$

But the equality of their normalized reports does not collapse their source identities.
Therefore:

two complementary constructions may both report $\frac{1}{2}$ without becoming the same construction.

This is the ratio-level version of the product distinction

$$(0, 1) \neq_{\mathrm{C}} (1, 0),$$

despite equal scalar projection.

A half report is consequently compatible with two opposing orientations, sources, or placements. The scalar value records their equal measure relative to the whole. It does not preserve which side was selected.

46 Reflection of a balanced carrier

Let

$$B \cong A \sqcup C$$

be balanced.
Define a reflection map

$$\rho_B : A \longrightarrow C$$

that pairs each occurrence of A with one occurrence of C.
Because $\mid A \mid = \mid C \mid$, ρ_B may be chosen as a bijection.
The reflection exchanges the complementary ratio reports:

$$\rho_B : \mathcal{R}(A, B) \longleftrightarrow \mathcal{R}(C, B).$$

Both project to

$$\frac{1}{2}.$$

The integer balance lane changes sign under exchange:

$$H(A; B) = \mid A \mid - \mid C \mid,$$

while

$$H(C; B) = \mid C \mid - \mid A \mid = -H(A; B).$$

At balance:

$$H(A; B) = 0.$$

Thus the half is the fixed scalar report of complementary reflection, while the integer lane exposes which side is larger before balance is reached.

47 No primitive critical half

The number $\frac{1}{2}$ need not be inserted into a construction before balance exists. The primitive sequence is

$$\text{whole denominator carrier } B,$$
$$\text{whole subreport } A,$$
$$\text{whole complement } C,$$
$$B \cong A \sqcup C,$$
$$|A| - |C| = 0,$$

followed by the normalized coordinate report

$$\frac{|A|}{|B|} = \frac{1}{2}.$$

Therefore,

$$\boxed{\text{balance precedes the half.}}$$

The half reports balance. It does not create balance.

This will later permit every appearance of a critical half to be rewritten without a primitive fractional scalar:

$$\frac{A}{B} = \frac{1}{2}$$

becomes

$$2A - B = 0.$$

The first is a normalized coordinate report. The second is the retained integer closure condition.

48 The primitive ratio hierarchy

The construction developed in this part may be summarized as follows:

$$\begin{gathered}\text{whole occurrences}\\ \Downarrow\\ \text{source-preserving grouping}\\ \Downarrow\\ A \cong (Q \times B) \sqcup R\\ \Downarrow\\ (q, R; B)\\ \Downarrow\\ \mathcal{R}(A, B)\\ \Downarrow\\ \frac{A}{B}.\end{gathered}$$

From this hierarchy follow:

1. the denominator is part of every ratio object;
2. a proper ratio retains the entire dividend as remainder;
3. a fraction does not require a fractional occurrence;
4. equivalent fractions are equal under cross-product balance;
5. scaling is common replication;
6. cancellation removes a proven common carrier;
7. reciprocals are unit-to-denominator ratios;
8. division by zero fails because no positive divisor carrier exists;
9. fraction arithmetic is common-carrier construction;
10. the half is the normalized report of two equal complementary carriers;

11. the integer balance condition

$$2A - B = 0$$

is prior to the coordinate report

$$A/B = 1/2.$$

Part VII
Finite Positional Reports

49 Finite positional notation and base expansion

Positional notation must be derived after counting, multiplication, division, and remainder. A written digit string is not the number prior to those constructions. It is an indexed report generated by repeated quotient–remainder extraction relative to a declared base carrier.

49.1 Base carrier and digit carrier

Definition 49.1 (Base carrier)**.** *A* base carrier B_b *is a positive finite occurrence report whose count is the base*

$$b = |B_b|, \qquad b \geq 2.$$

The condition $b \geq 2$ ensures that repeated extraction can distinguish at least a null digit report from one positive digit report. The base is not merely a subscript on a numeral. It is part of the positional grammar.

Definition 49.2 (Digit report)**.** *A* digit report in base b *is a remainder-count report* d *satisfying*

$$0 \leq d < b.$$

Constructively, d *reports the remainder carrier left after extracting complete copies of* B_b *from a current dividend state.*

Thus a digit is not primitively chosen from a written alphabet. Its allowed range is derived from the remainder condition.

49.2 Repeated quotient–remainder extraction

Let N_0 be a finite occurrence report with count n_0. Divide by the base carrier:

$$n_0 = q_0 b + d_0, \qquad 0 \leq d_0 < b.$$

The remainder d_0 is the least-position digit report. The quotient carrier becomes the next dividend:

$$q_0 = q_1 b + d_1.$$

Continue recursively:

$$q_k = q_{k+1} b + d_{k+1}, \qquad 0 \leq d_{k+1} < b.$$

Because the initial report is finite and each positive quotient is smaller than its preceding positive dividend when $b \geq 2$, the process terminates at some stage m with

$$q_m = 0.$$

Definition 49.3 (Finite positional ledger)**.** *The* finite positional ledger *of N_0 in base b is the ordered remainder sequence*

$$\Lambda_b(N_0) = [d_m, d_{m-1}, \ldots, d_1, d_0],$$

together with the retained base, digit indices, quotient chain, and termination zero.

The displayed numeral

$$d_m d_{m-1} \cdots d_1 d_{0b}$$

is a compressed projection of this ledger.

Theorem 49.4 (Finite base-expansion theorem)**.** *Every finite natural-number report N_0 and every declared base $b \geq 2$ generate a terminating quotient–remainder ledger. The resulting digit sequence reconstructs the original count uniquely when the base and digit positions are retained.*

Proof. At each step, quotient–remainder uniqueness gives unique q_k and d_k. The quotient chain strictly descends while positive, so it terminates. Substituting the equations backward reconstructs the initial count. Uniqueness follows recursively from uniqueness at each division step. □

49.3 Place weights without primitive powers

Define the first place carrier as the unit report

$$W_0 = U.$$

Define each later place carrier recursively by repeated base product:

$$W_{k+1} = B_b \times W_k.$$

Its count is

$$|W_{k+1}| = b|W_k|.$$

Only after repeated multiplication has been established may this count be abbreviated as

$$|W_k| = b^k.$$

Thus exponent notation is a compressed report of a recursively constructed place carrier.

The complete reconstruction of the numeral ledger is

$$N_0 \cong \bigsqcup_{k=0}^{m} (D_k \times W_k),$$

where D_k is a digit carrier of count d_k. Its scalar projection is

$$n_0 = \sum_{k=0}^{m} d_k b^k.$$

The sum and powers are not primitive ingredients; they summarize the retained place-product ledger.

49.4 A zero digit is an indexed placeholder

A zero digit does not mean that the report position is absent. It means that no positive copy of the corresponding place carrier contributes at that retained index.

For example, the base-two report

$$[1, 0, 1]_2$$

contains three digit positions. The middle position is occupied by a constructive zero digit reporting no 2^1-place contribution. Removing that position would change the positional report:

$$[1, 0, 1]_2 \neq_{\mathrm{C}} [1, 1]_2.$$

Their reconstructed counts are different because the remaining 1 digits are attached to different place carriers.

Theorem 49.5 (Positional-zero theorem)**.** *Whenever a zero digit occurs between retained positional digits, it is a whole indexed report object preserving a place relation. It cannot be replaced by literal absence without changing the positional ledger.*

Leading zero digits require a declared width or boundary to remain significant. Thus

$$[0, 1]_2$$

and

$$[1]_2$$

may share a normalized scalar count while remaining different fixed-width reports. Normalization that deletes leading zeros is another projection and must be declared.

49.5 Base dependence and report equality

The same constructive count may admit different positional reports in different bases. For example,

$$[1, 0]_2, \qquad [2]_{10},$$

report the same count but use different base carriers, digit alphabets, and place ledgers.

Therefore

$$[1, 0]_2 =_{\mathrm{count}} [2]_{10}$$

does not imply full constructive identity. Positional equality is typed by at least:

(a) the base carrier;

(b) the ordered digit positions;

(c) the digit values;

(d) any retained width or leading-zero boundary;

(e) the radix boundary, when present;

(f) and the quotient–remainder construction history if that history is relevant.

49.6 Radix expansion of a proper ratio

Let a proper constructive ratio have numerator count A and positive denominator count D:

$$0 < A < D.$$

Its initial quotient–remainder state is

$$A = 0 \cdot D + A.$$

To generate a base-b positional report after the radix boundary, replicate the current remainder by the base carrier and divide by the retained denominator:

$$bR_k = d_{k+1}D + R_{k+1}, \qquad 0 \le R_{k+1} < D.$$

The quotient d_{k+1} is the next digit report, and the remainder R_{k+1} carries the unresolved ratio obligation into the next position.

Definition 49.6 (Finite radix-state prefix)**.** *After m recurrence steps, the finite radix-state prefix is*

$$0.d_1 d_2 \cdots d_{mb}$$

together with the retained remainder R_m and denominator D.

The written prefix alone is not the complete ratio. The exact finite state is

$$(d_1, \ldots, d_m; R_m, D, b).$$

49.7 Examples of base-dependent ratio reports

For the ratio $1/2$ in base two, begin with remainder $R_0 = 1$ and denominator $D = 2$:

$$2R_0 = 2 = 1 \cdot 2 + 0.$$

Thus the first digit is 1 and the remainder closes:

$$\frac{1}{2} = 0.1_2.$$

For the same ratio in base ten,

$$10R_0 = 10 = 5 \cdot 2 + 0,$$

so

$$\frac{1}{2} = 0.5_{10}.$$

The ratio construction is the same; the positional reports differ because the base carriers differ.

For $1/3$ in base ten,

$$10 \cdot 1 = 3 \cdot 3 + 1.$$

The next remainder is again 1, so the same state recurs and the digit 3 repeats. The exact constructive object remains the ratio $\mathcal{R}(1, 3)$ together with a one-state decimal remainder cycle. The notation

$$0.333\ldots_{10}$$

is a report of that recurring state, not an infinite pile of already completed written digits.

49.8 Termination and recurrence

Theorem 49.7 (Termination criterion)**.** *A radix expansion terminates exactly when some generated remainder is the typed exact-remainder zero:*

$$R_m = 0_{A:D,b}.$$

Theorem 49.8 (Finite remainder recurrence)**.** *For a fixed positive denominator D, every nonzero remainder satisfies*

$$1 \leq R_k < D.$$

There are only finitely many possible positive remainder counts. Therefore a nonterminating radix recursion must eventually revisit a prior remainder state, after which its digit and remainder sequence recurs under the same deterministic division rule.

Proof. The remainder condition restricts each state to one of $D - 1$ positive counts. If no zero state is reached, a deterministic sequence with more than $D - 1$ steps must repeat a prior state. Equal current remainder, denominator, and base produce equal next quotient and remainder by division uniqueness. □

Thus every rational ratio has either a terminating or recurring positional ledger in a finite base grammar. The recurrence is a finite-state property of retained remainders.

49.9 The radix point is a report boundary

The radix point is not a geometric point inserted into the counted object. It is a report marker separating complete base-place extractions from recursively scaled residual-place reports.

To the left of the radix boundary, positions report complete place carriers. To the right, positions report repeated base scaling of an unresolved remainder relative to the retained denominator. The boundary is part of the notation grammar, not a primitive location in reality.

49.10 Finite prefixes are not completed limits

A finite prefix of a recurring expansion retains a nonzero remainder. Therefore it is not constructively identical to a completed scalar that a later real-number grammar may associate with the full recurrence.

For example, each finite decimal prefix

$$0.9, \quad 0.99, \quad 0.999, \quad \ldots$$

retains a positive unresolved remainder relative to its finite place carrier. No finite prefix is constructively equal to the complete unit report. Any later statement identifying an indefinitely recurring positional rule with 1 must specify a completion or limit operation and prove what information that operation preserves.

This manuscript has not yet introduced such a completion. It therefore retains the distinction among:

(1) a finite digit prefix;

(2) its current remainder state;

(3) a finite recurring-state description;

(4) and a later completed real-number report.

49.11 Positional-report preservation theorem

Theorem 49.9 (Positional-report preservation)**.** *A positional numeral is a lossless report of its finite quotient–remainder construction only when the following are retained:*

(i) the base carrier;

(ii) digit order and whole digit indices;

(iii) the value of every digit, including zero placeholders;

(iv) the radix boundary when present;

(v) the termination zero or current remainder;

(vi) and any fixed-width or normalization rule relevant to identity.

If any of these are discarded, equal displayed values need not imply equal constructive reports.

Before completion and limits can be stated without circularity, the next section constructs signed orientation, negative-number reports, additive inverses, and signed differences. A later completion section must then distinguish stabilization of a finite remainder rule from the unsupported claim that fixed positive primitives literally become zero.

Part VIII
Signed Orientation and Integer Reports

50 Signed orientation, negative numbers, and additive inverse

Subtraction has so far been constructed as removal when the occurrences identified for removal are already contained in the source report. That construction is sufficient for natural-number subtraction with a nonnegative remainder, but it does not yet define what is reported when the requested removal exceeds the available positive carrier.

A negative report cannot be constructed by claiming that a set contains fewer than no occurrences. Every admitted occurrence remains whole and positively countable as an occurrence. What changes is not the existence of the occurrence but the *orientation of its obligation.* Signed arithmetic therefore requires two retained lanes: a contribution lane and an opposing obligation lane.

50.1 Declared unit equivalence

Signed cancellation requires a criterion specifying which opposing occurrences may discharge one another.

Definition 50.1 (Count-unit equivalence)**.** *A* count-unit equivalence *is a declared relation $\equiv_1$ under which one whole occurrence in one orientation lane may be matched with one whole occurrence in the opposing lane.*

For untyped integer counting, every admitted counting occurrence is equivalent under $\equiv_1$. For typed constructions, cancellation may be narrower: an occurrence can discharge only an opposing occurrence of the declared matching type. The matching criterion is therefore part of the signed construction and may not be omitted when source identity matters.

50.2 Oriented two-lane state

Definition 50.2 (Oriented count state)**.** *An* oriented count state *is a report*

$$\mathsf{Z}(P, N; \gamma),$$

where:

(i) P *is the positive-orientation occurrence carrier or a typed zero-lane marker;*

(ii) N *is the negative-orientation occurrence carrier or a typed zero-lane marker;*

(iii) γ *records the source relation, unit-equivalence criterion, and any cancellation history declared relevant.*

The two lanes are not two scalar coordinates assumed in advance. They are two distinguishable roles in one report:

$$\text{retained contribution} \qquad \text{and} \qquad \text{retained opposing obligation.}$$

The state $\mathsf{Z}(P, N; \gamma)$ is therefore prior to the conventional notation

$$|P| - |N|.$$

That notation is the later signed-count projection of the retained two-lane construction.

Principle 50.3 (No negative multiplicity)**.** *A negative integer never reports a negative number of existing occurrences. It reports a positive whole-occurrence carrier retained in the opposing orientation lane.*

50.3 Pair cancellation and signed normal form

Definition 50.4 (Opposing pair)**.** *An* opposing pair *is a pair* $(p, n) \in P \times N$ *satisfying*

$$p \equiv_1 n.$$

It records one positive occurrence and one negative occurrence that may discharge one another under the declared signed grammar.

Definition 50.5 (Signed cancellation)**.** *A* signed cancellation *is a source-preserving matching*

$$M \subseteq P \times N$$

whose pairs are occurrence-disjoint in both lanes. Removing every matched occurrence from both lanes produces residual carriers

$$P_M = P \setminus \mathrm{dom}(M), \qquad N_M = N \setminus \mathrm{im}(M).$$

The complete cancellation report retains

$$(P, N, M, P_M, N_M).$$

The set-minus notation here reports source-preserving removal already constructed in Section 10; it does not assert literal disappearance. Every matched pair is retained in the cancellation ledger even when neither member survives in the residual magnitude report.

Definition 50.6 (Maximal signed cancellation)**.** *A signed cancellation is* maximal *when no further admissible opposing pair can be formed from P_M and N_M.*

Theorem 50.7 (One-lane normal form for pure counts)**.** *Under universal count-unit equivalence, maximal cancellation leaves at most one positive residual lane:*

$$P_M \neq 0 \Longrightarrow N_M = 0, \qquad N_M \neq 0 \Longrightarrow P_M = 0.$$

Proof. Suppose both residual lanes contained a positive occurrence. Universal count-unit equivalence would make those two occurrences an admissible opposing pair, contradicting maximality. Therefore at least one residual lane must carry zero positive contribution. □

Definition 50.8 (Signed normal form)**.** *The* signed normal form *of $\mathsf{Z}(P, N; \gamma)$ is the residual oriented state*

$$\mathrm{Norm}_\gamma(P, N) = \mathsf{Z}(P_M, N_M; \gamma_M)$$

produced by a maximal declared cancellation.

For pure finite counts, the residual magnitude is unique at the count level even when the particular matched source pairs are not. Different cancellation matchings may therefore share one signed scalar report while retaining different source ledgers.

50.4 Positive, negative, and neutral orientation

Definition 50.9 (Orientation report)**.** *Let $\mathrm{Norm}_\gamma(P, N) = \mathsf{Z}(P', N'; \gamma')$. Its orientation is reported as follows:*

$$\mathrm{sgn}_{\mathrm{C}}(P, N) = \begin{cases} +, & P' \text{ has positive count and } N' \text{ is a zero lane,} \\ -, & N' \text{ has positive count and } P' \text{ is a zero lane,} \\ 0, & P' \text{ and } N' \text{ are both zero lanes.} \end{cases}$$

The signs + and − are orientation tokens. They do not create the carriers they report. The token 0 reports neutralized orientation after all admissible opposing pairs have closed.

Definition 50.10 (Magnitude report)**.** *The* magnitude *of a normalized oriented state is the positive count of its surviving lane. A neutral state has magnitude zero.*

Thus a normalized signed report separates two questions:

which orientation survives? and how many whole occurrences survive there?

50.5 Construction of the integers

Let A_n be any n-occurrence count carrier, and let 0_+ and 0_- denote typed zero-lane markers preserving the positive and negative lane positions.

Definition 50.11 (Positive integer report)**.** *The positive integer report associated with A_n is*

$$+n := \mathsf{Z}(A_n, 0_-).$$

Definition 50.12 (Negative integer report)**.** *The negative integer report associated with A_n is*

$$-n := \mathsf{Z}(0_+, A_n).$$

Definition 50.13 (Neutral integer report)**.** *A normalized neutral integer report is*

$$0_{\mathbb{Z}} := \mathsf{Z}(0_+, 0_-).$$

The report $-n$ therefore means:

n whole count occurrences remain in the opposing orientation lane.

It does not mean that n objects have become nonobjects or that reality contains a negative multiplicity.

Theorem 50.14 (Natural-number embedding)**.** *The map*

$$\iota : \mathbb{N} \longrightarrow \mathbb{Z}_{\mathrm{C}}, \qquad n \longmapsto \mathsf{Z}(A_n, 0_-)$$

preserves natural-number count equality and source-preserving addition at the signed-report level.

Here $\mathbb{Z}_{\mathrm{C}}$ denotes the constructed family of oriented count states, not a number line assumed before the construction.

50.6 Integer-report equivalence

Different two-lane ledgers may produce the same normalized signed count. A precise equality relation is therefore required.

Definition 50.15 (Integer-report equivalence)**.** *Two oriented states are integer-report equivalent,*

$$\mathsf{Z}(P, N) \sim_{\mathbb{Z}} \mathsf{Z}(P', N'),$$

when the cross-joined natural carriers have equal size:

$$P \sqcup N' \ =_{\text{size}} \ P' \sqcup N.$$

Equivalently at the count level,

$$|P| + |N'| = |P'| + |N|.$$

This definition uses only natural occurrence counts and source-preserving joins. It does not presuppose signed subtraction.

Theorem 50.16 (Integer-report equivalence is an equivalence relation)**.** *The relation $\sim_{\mathbb{Z}}$ is reflexive, symmetric, and transitive.*

Proof. Reflexivity follows because $P \sqcup N$ has the same size as itself. Symmetry follows by reversing the size equality. For transitivity, suppose

$$|P| + |N'| = |P'| + |N|$$

and

$$|P'| + |N''| = |P''| + |N'|.$$

Joining the equalities and cancelling the common natural carrier represented by P' and N' yields

$$|P| + |N''| = |P''| + |N|.$$

The cancellation is legitimate because the same declared replication or join carrier occurs on both sides. □

Remark 50.17 (Equivalence is not source identity)**.** *From*

$$\mathsf{Z}(P, N) \sim_{\mathbb{Z}} \mathsf{Z}(P', N')$$

it does not follow that the two ledgers are constructively identical. The relation certifies equality of their signed count reports. Source, type, route, and cancellation history remain separate unless a further preservation theorem permits their erasure.

50.7 Signed addition

Definition 50.18 (Lane-wise signed join)**.** *For oriented states*

$$X = \mathsf{Z}(P, N), \qquad Y = \mathsf{Z}(Q, R),$$

define their unnormalized signed join by

$$X \boxplus Y := \mathsf{Z}(P \sqcup Q,\ N \sqcup R).$$

Their signed sum is its declared normal form:

$$X \oplus Y := \mathrm{Norm}(X \boxplus Y).$$

Positive contributions join positive contributions; negative obligations join negative obligations; and only then are opposing lanes cancelled. This order preserves source before scalar normalization.

Theorem 50.19 (Signed-addition report)**.** *If X and Y have conventional signed-count reports x and y, then $X \oplus Y$ reports $x + y$.*

Proof. Before cancellation, the positive lane contains the natural join $P \sqcup Q$ and the negative lane contains $N \sqcup R$. Every opposing cancellation removes one occurrence from each lane and therefore preserves their cross-lane imbalance. The surviving orientation and magnitude consequently report the sum of the two signed imbalances. □

Theorem 50.20 (Constructive commutativity and associativity of signed addition)**.** *Signed addition is commutative and associative up to source-preserving reindexing, reassociation, and integer-report equivalence.*

Proof. Commutativity follows from commutativity of the two source-preserving lane joins up to reindexing. Associativity follows from reassociation of those joins. Normalization may choose different matched source pairs, but every maximal cancellation preserves the same signed count report. □

50.8 Additive inverse

Definition 50.21 (Orientation reversal)**.** *The* orientation reversal *of*

$$X = \mathsf{Z}(P, N; \gamma)$$

is

$$-X := \mathsf{Z}(N, P; \gamma^{-}),$$

where γ^{-} records that the lane roles have been exchanged.

Orientation reversal does not destroy or recreate occurrences. It exchanges which lane reports contribution and which reports opposing obligation.

Theorem 50.22 (Additive inverse closure)**.** *For every oriented state X,*

$$X \oplus (-X) \sim_{\mathbb{Z}} 0_{\mathbb{Z}}.$$

Proof. If $X = \mathsf{Z}(P, N)$, then

$$X \boxplus (-X) = \mathsf{Z}(P \sqcup N,\ N \sqcup P).$$

Transposition and reindexing provide a size-preserving pairing between the two lanes. Maximal cancellation therefore leaves typed zero lanes. □

The complete closure ledger may still retain every source pair that cancelled. Scalar zero does not erase the construction that produced it.

50.9 Signed subtraction

Definition 50.23 (Signed subtraction)**.** *For oriented states X and Y, define*

$$X \ominus_{\mathbb{Z}} Y := X \oplus (-Y).$$

This identity is now derived rather than assumed. Removal-based subtraction was constructed first; orientation reversal and signed addition were then constructed; only now may subtraction be extended beyond natural containment by addition of an inverse.

Theorem 50.24 (Extension of natural subtraction)**.** *Let A and B be natural occurrence carriers embedded as positive oriented states. Then:*

(i) if $B \preceq_{\mathrm{C}} A$, the state $\iota(A) \ominus_{\mathbb{Z}} \iota(B)$ normalizes to the positive natural remainder already constructed by removal;

(ii) if $A \preceq_{\mathrm{C}} B$, it normalizes to the negative orientation of the natural remainder obtained by removing A from B;

(iii) if A and B have equal size, it normalizes to an indexed zero closure.

Thus a negative result is a retained excess obligation, not failed arithmetic and not a set with negative cardinality.

50.10 Signed zero and balanced ledgers

A nonnormalized state may carry equal positive and negative lanes:

$$\mathsf{Z}(K, K).$$

It is not constructively identical to a bare normalized zero report because it retains the source carrier K twice and the relation pairing its opposing occurrences. Nevertheless,

$$\mathsf{Z}(K, K) \sim_{\mathbb{Z}} 0_{\mathbb{Z}}.$$

Theorem 50.25 (Balanced-pair zero)**.** *An oriented state reports integer zero exactly when its positive and negative lanes are size-equal:*

$$\mathsf{Z}(P, N) \sim_{\mathbb{Z}} 0_{\mathbb{Z}} \iff |P| = |N|.$$

Proof. By integer-report equivalence with $\mathsf{Z}(0_+, 0_-)$,

$$|P| + 0 = 0 + |N|,$$

which is precisely size equality of the two lanes. □

This is the signed form of the earlier balance law. Zero is the closure report of two opposing equal carriers, while the carriers and their cancellation lineage may remain available in the full ledger.

Remark 50.26 (Signed representations of zero)**.** *A notation system may retain tokens such as* $+0$ *and* -0*. Those tokens are distinct oriented* reports *if the sign position is preserved, but they represent the same normalized integer magnitude and orientation-neutral closure. A later projection may identify them only after declaring that the retained sign token is irrelevant to the property under study.*

50.11 Order on signed reports

Signed order is derived from oriented difference, not from a pre-existing number line.

Definition 50.27 (Signed comparison)**.** *For oriented states* X *and* Y*, define*

$$X \preceq_{\mathbb{Z}} Y$$

when the normalized difference

$$Y \ominus_{\mathbb{Z}} X$$

has positive or neutral orientation. Define $X \prec_{\mathbb{Z}} Y$ *when that difference has strictly positive orientation.*

Theorem 50.28 (Natural-order preservation)**.** *The signed comparison restricted to positively embedded natural carriers agrees with constructive containment comparison.*

Proof. For positive embeddings, $\iota(B) \ominus_{\mathbb{Z}} \iota(A)$ places B in the positive lane and A in the negative lane. It normalizes to positive or zero exactly when every occurrence of A can be matched within B. □

The conventional ordering

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$$

is therefore a later linear display of already constructed orientation, magnitude, and comparison relations.

50.12 Signed multiplication

Multiplication of oriented states must retain whether source orientations agree or oppose.

Definition 50.29 (Oriented product ledger)**.** *Let*

$$X = \mathsf{Z}(P, N), \qquad Y = \mathsf{Z}(Q, R).$$

Define the unnormalized oriented product by

$$X \boxtimes Y := \mathsf{Z}\Big((P \times Q) \sqcup (N \times R), (P \times R) \sqcup (N \times Q)\Big).$$

Its signed product is

$$X \otimes_{\mathbb{Z}} Y := \mathrm{Norm}(X \boxtimes Y).$$

The positive lane receives products whose orientations agree. The negative lane receives products whose orientations oppose. This is a retained four-route product ledger, not merely a memorized sign table.

Theorem 50.30 (Constructive sign rules)**.** *For positive magnitudes a and b, oriented multiplication yields*

$$(+a)(+b) = +(ab),$$
$$(+a)(-b) = -(ab),$$
$$(-a)(+b) = -(ab),$$
$$(-a)(-b) = +(ab).$$

Proof. For two positive states, only $P \times Q$ is nonzero, so the product is positive. For a positive and a negative state, only one cross-orientation product lane survives, so the product is negative. For two negative states, the product $N \times R$ enters the agreeing-orientation lane and the opposing lane is zero, so the product is positive. □

The rule

$$(-a)(-b) = ab$$

is therefore not an arbitrary convention. Reversing orientation twice returns the product to the agreeing lane.

Theorem 50.31 (Zero absorption)**.** *For every oriented state X,*

$$X \otimes_{\mathbb{Z}} 0_{\mathbb{Z}} \sim_{\mathbb{Z}} 0_{\mathbb{Z}}.$$

Proof. Both product lanes contain only typed no-copy products because both lanes of normalized integer zero have zero contribution. The result is a typed multiplication-closure zero. □

Theorem 50.32 (Signed distributivity)**.** *Oriented multiplication distributes over oriented addition up to source-preserving product decomposition, lane reindexing, and integer-report equivalence.*

Proof. Expand each product lane using the previously proved constructive distribution law. The positive and negative subledgers then regroup into the same source families on both sides. Any difference in cancellation order changes only the retained cancellation route, not the integer-report equivalence class. □

50.13 The sign token and positional notation

A sign token is not a radix digit. It applies an orientation report to an already constructed magnitude ledger.

If

$$d_m d_{m-1} \cdots d_{0b}$$

is a finite positional magnitude report, then

$$-d_m d_{m-1} \cdots d_{0b}$$

means that the magnitude carrier reconstructed by the digit ledger is placed in the negative orientation lane.

Every digit remains a nonnegative remainder report satisfying

$$0 \leq d_k < b.$$

The leading minus sign does not make the digits negative and does not change their place carriers. It changes the orientation of the completed magnitude report.

Theorem 50.33 (Signed positional preservation)**.** *A signed positional numeral is lossless relative to its finite signed construction only when it retains:*

(i) the orientation token or neutral orientation;

(ii) the magnitude's base carrier;

(iii) all digit indices and digit values;

(iv) the radix boundary, width, and zero placeholders when relevant;

(v) and the source relation by which the magnitude was assigned to that orientation.

Deleting the sign token is therefore a projection, just as deleting a zero placeholder or base marker is a projection.

50.14 Orientation precedes the number line

Signed orientation requires only opposing report roles, whole occurrences, matching, and cancellation. It does not require a Cartesian axis or a continuous line.

A conventional number line may later choose:

(i) one marked origin to report normalized zero;

(ii) one direction to report positive orientation;

(iii) the opposite direction to report negative orientation;

(iv) and repeated equal placement intervals to report magnitude.

The line is therefore a geometric projection of the signed order already constructed. It does not create negative numbers.

Principle 50.34 (Orientation before coordinate direction)**.** *Opposition is constructively prior to left and right. A coordinate convention may display the positive lane to the right and the negative lane to the left, but that choice is not part of the primitive signed relation.*

50.15 Why signed orientation precedes limits

Limits, error terms, upper and lower bounds, oscillation, and convergence all require the ability to distinguish opposing difference states. Before one can report that a sequence lies above or below a candidate closure value, or that an error changes orientation while its magnitude contracts, one must already possess:

(i) signed difference;

(ii) additive inverse;

(iii) neutral cancellation;

(iv) signed order;

(v) and magnitude independent of orientation.

Completion cannot be used to construct these notions without circularity, because conventional definitions of completion already compare signed differences between finite reports.

50.16 Signed-orientation preservation theorem

Theorem 50.35 (Signed-orientation preservation)**.** *A scalar integer report preserves its constructive signed source only when the reporting map retains, or proves irrelevant, all of the following:*

(i) the positive and negative source carriers;

(ii) the declared unit-equivalence criterion;

(iii) the cancellation matching or a theorem of matching-independence;

(iv) the surviving orientation;

(v) the surviving whole-occurrence magnitude;

(vi) and every typed zero closure relevant to the operation.

Equal normalized integers do not by themselves imply equal signed construction ledgers.

The primitive dependency is now

$$\begin{aligned}\text{natural occurrence carriers} &\longrightarrow \text{opposing lanes} \longrightarrow \text{pair cancellation}\\ &\longrightarrow \text{orientation and magnitude} \longrightarrow \text{integer report}\\ &\longrightarrow \text{signed arithmetic.}\end{aligned}$$

Only after this signed layer exists can the manuscript construct powers with negative exponents, signed error states, two-sided bounds, and eventually limits and completion without silently borrowing the integer system it is meant to explain.

Part IX
Powers, Roots, and Operational Depth

51 Powers, repeated composition, and root reconstruction

Multiplication constructs an ordered placement product from two declared carriers. A power requires a further relation: the same declared carrier is repeatedly composed with itself under a retained depth index. The scalar notation a^n is therefore not primitive. It is the count projection of a recursively constructed product ledger whose base, depth, grouping, orientation, and zero-depth convention remain part of the complete report.

This section constructs finite powers before any appeal to infinite series, exponential functions, irrational roots, limits, or completion. It also constructs exact root states and finite root-shell residuals. Non-exact roots that require a completed real carrier are deliberately postponed.

51.1 Power depth and repeated self-product

Let $A = (I_A, v_A)$ be a finite occurrence report and let K_n be a finite depth report containing n whole depth indices.

Definition 51.1 (Positive-depth power ledger)**.** *For positive depth, define the power ledger recursively by*

$$\mathrm{Pow}_1(A) = A,$$

and

$$\mathrm{Pow}_{n+1}(A) = \mathrm{Pow}_n(A) \times A.$$

An occurrence of $\mathrm{Pow}_n(A)$ *is an ordered source tuple*

$$(a_{i_1}, a_{i_2}, \ldots, a_{i_n}),$$

with one whole occurrence selected from A at every retained depth position.

The depth positions are not copies of a scalar exponent. They are report indices selecting how many self-product stages are retained. Their order records which factor role each occurrence occupies.

Definition 51.2 (Power source address)**.** *The complete source address of a positive-depth power occurrence is*

$$(k_1, i_1;\ k_2, i_2;\ \ldots;\ k_n, i_n),$$

where k_j is the jth depth role and i_j selects an occurrence from A.

After the recursive construction has been established, the product index carrier may be abbreviated by

$$I_A^n$$

for the n-fold ordered product of I_A with itself. The notation does not create the product; it names the already constructed depth ledger.

Theorem 51.3 (Finite power count)**.** *If A has constructive count a and K_n has positive count n, then*

$$\mathrm{Count}(\mathrm{Pow}_n(A)) = a^n,$$

where a^n abbreviates the count of the n-fold self-product ledger.

Proof. At depth one, $\mathrm{Pow}_1(A) = A$, so the count is a. Suppose the depth-n ledger has count a^n. The next ledger is its placement product with A, so multiplication gives

$$\mathrm{Count}(\mathrm{Pow}_{n+1}(A)) = \mathrm{Count}(\mathrm{Pow}_n(A))\, a = a^n a.$$

The scalar abbreviation for this recursively repeated product is a^{n+1}. □

Thus the dependency is

base carrier $\longrightarrow$ depth carrier $\longrightarrow$ ordered self-product $\longrightarrow$ power ledger $\longrightarrow$ scalar power report.

51.2 Depth one and depth zero

Depth one retains exactly one factor role, so

$$\mathrm{Pow}_1(A) = A$$

and

$$a^1 = a.$$

Depth zero requires care. It cannot be literal absence, because a zero-depth operation must still be distinguishable as an operation report.

Definition 51.4 (Zero-depth product unit)**.** *The zero-depth power ledger of A is the typed singleton*

$$\mathrm{Pow}_0(A) = U_{\times,A} = [\varepsilon_A],$$

where ε_A reports one completed empty factor selection: no factor occurrence was requested, and the product obligation is closed at depth zero.

The object ε_A is not an occurrence of A. It is a typed product-identity report. Its count is one:

$$\mathrm{Count}(U_{\times,A}) = 1.$$

Therefore the finite fold convention reports

$$a^0 = 1.$$

Theorem 51.5 (Zero-depth identity)**.** *For every declared finite base report A,*

$$\mathrm{Pow}_0(A) \times \mathrm{Pow}_n(A) \cong \mathrm{Pow}_n(A).$$

Proof. The zero-depth carrier contains one identity occurrence. Pairing every depth-n tuple with that single occurrence changes no choice count and admits a canonical normalization that removes the typed identity role. □

Remark 51.6 (The underspecified symbol 0^0)**.** *Inside the finite product-fold grammar, a zero-depth fold has one empty tuple, so a typed expression*

$$\mathrm{Pow}_0(0_A)$$

reports the multiplicative unit. A bare expression 0^0 may also be used in later analytic, limiting, functional, or combinatorial grammars with different boundary requirements. Those grammars are not automatically identical. Therefore the untyped symbol 0^0 has no universal interpretation merely from its typography; the governing operation and domain must be declared.

For every positive depth n,

$$\mathrm{Pow}_n(0_A) = 0_{\mathrm{Pow}_n(A)},$$

because a positive-depth tuple requires a positive selection from the base at every factor role, and the zero base supplies no positive contribution.

51.3 Concatenation, flattening, and the finite exponent laws

The familiar exponent laws arise from explicit transformations of depth tuples.

Theorem 51.7 (Depth concatenation)**.** *For nonnegative depths m and n,*

$$\mathrm{Pow}_{m+n}(A) \cong \mathrm{Pow}_m(A) \times \mathrm{Pow}_n(A).$$

Consequently,

$$a^{m+n} = a^m a^n.$$

Proof. A depth-$(m+n)$ tuple has an initial block of m factor roles and a final block of n factor roles. Splitting at that retained boundary gives one depth-m tuple paired with one depth-n tuple. Concatenation reverses the map. The two ledgers are therefore in bijective, order-preserving correspondence. □

Theorem 51.8 (Depth flattening)**.** *For nonnegative depths m and n,*

$$\mathrm{Pow}_n(\mathrm{Pow}_m(A)) \cong \mathrm{Pow}_{mn}(A).$$

Consequently,

$$(a^m)^n = a^{mn}.$$

Proof. An occurrence on the left is an ordered family of n blocks, each containing m ordered selections from A. Flattening the block addresses gives one ordered tuple with mn factor roles. Restoring the block boundaries reverses the construction. □

Theorem 51.9 (Product power decomposition)**.** *For nonnegative depth n,*

$$\mathrm{Pow}_n(A \times B) \cong \mathrm{Pow}_n(A) \times \mathrm{Pow}_n(B).$$

Consequently,

$$(ab)^n = a^n b^n.$$

Proof. Each depth role on the left selects an ordered pair (a_i, b_j). Unzipping the n selected pairs gives one ordered A-tuple and one ordered B-tuple. Zipping the corresponding depth positions reverses the map. □

These are correspondence theorems among retained ledgers. They do not state that the differently grouped source syntax is literally identical.

51.4 Equal scalar powers and unequal power constructions

Power projection can collapse both base and depth information.

Example 51.10 (Equal value, different depth ledger)**.** *The scalar reports*

$$2^4 = 16 \qquad \textit{and} \qquad 4^2 = 16$$

are equal, but their construction ledgers are different. The first contains four binary factor roles. The second contains two four-choice factor roles. A value-only projection identifies their counts while discarding base carrier, depth carrier, tuple arity, and grouping history.

Even more sharply,

$$1^1 = 1^2 = 1^3 = \cdots = 1$$

at the scalar level, while the corresponding power ledgers retain different numbers of typed factor roles. A normalization may prove those roles irrelevant for a particular property, but scalar equality alone does not do so.

Principle 51.11 (Power projection)**.** *From*

$$a^m = b^n$$

one may infer equal scalar power reports. One may not infer equal bases, equal depths, equal tuple ledgers, or equal operational histories without an additional preservation theorem.

51.5 Signed bases and parity of depth

Signed exponentiation is repeated signed multiplication. Every occurrence remains positively countable as an occurrence; the sign reports the orientation of the resulting product lane.

Definition 51.12 (Parity report)**.** *For a nonnegative depth count n, divide by the two-occurrence carrier:*

$$n = 2q + r, \qquad r \in \{0, 1\}.$$

The depth is even *when $r = 0$ and* odd *when $r = 1$.*

Parity is therefore a quotient–remainder report, not a primitive label attached to the numeral.

Theorem 51.13 (Orientation of a finite signed power)**.** *Let X be a normalized nonzero signed integer report and let $n > 0$.*

(i) If X is positively oriented, then X^n is positively oriented.

(ii) If X is negatively oriented and n is even, then X^n is positively oriented.

(iii) If X is negatively oriented and n is odd, then X^n is negatively oriented.

Proof. Each multiplication of two equally negative orientations contributes to the positive product lane. An even depth partitions all negative factors into pairs. An odd depth leaves one unmatched negative orientation after all possible pairs have been formed. Positive factors never change the orientation. □

At the scalar level this yields the familiar reports

$$(-a)^{2q} = a^{2q}, \qquad (-a)^{2q+1} = -a^{2q+1}.$$

The right-hand signs are reports of the completed orientation ledger, not signs attached independently to each counted occurrence.

51.6 Negative exponents as reciprocal power reports

A negative exponent cannot mean a negative number of multiplication stages. The depth magnitude remains a nonnegative whole count. The negative orientation of the exponent instructs the report to form a reciprocal after the positive-depth power has been constructed.

Definition 51.14 (Signed reciprocal)**.** *Let X be a nonzero signed integer report with magnitude carrier M_X. Its signed reciprocal is the unit-to-magnitude ratio*

$$\mathrm{Rec}_{\mathbb{Z}}(X) = \mathrm{sgn}(X)\,\mathcal{R}(U, M_X),$$

where the ratio receives the same orientation as X.

Definition 51.15 (Negative power)**.** *For positive n and nonzero X, define*

$$X^{-n} = \mathrm{Rec}_{\mathbb{Z}}(X^n).$$

Equivalently,

$$X^{-n} = \frac{1}{X^n}$$

is the scalar report of that signed reciprocal relation.

Thus the minus sign in the exponent does not negate the product count. It changes the final operation from retained product to reciprocal-of-product.

Theorem 51.16 (Positive and negative depth closure)**.** *For nonzero X and positive n,*

$$X^n \cdot X^{-n} =_{\mathrm{ratio}} 1.$$

Proof. By definition, X^{-n} is the unit-to-X^n ratio with the appropriate orientation. Multiplying the magnitude carrier by its reciprocal gives an exact common-carrier cancellation, while the equal orientations multiply to positive orientation. □

Zero has no negative power because the construction would require the reciprocal of a zero-contribution denominator:

$$0^{-n} \quad \text{is undefined for } n > 0.$$

51.7 Powers of retained ratios

Let $\mathcal{R}(A, B)$ be a constructive ratio with positive denominator carrier B.

Definition 51.17 (Finite ratio power)**.** *For nonnegative depth n, define*

$$\mathcal{R}(A, B)^n = \mathcal{R}(\mathrm{Pow}_n(A), \mathrm{Pow}_n(B)).$$

Its scalar projection is

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

At depth zero, both numerator and denominator are typed unit carriers, giving the ratio report one. At positive depth, a zero numerator produces a zero ratio, while the denominator remains positive.

A reduction after powering may remove common replication carriers, but the reduced ratio does not by itself preserve the original base ratio or power depth.

51.8 Roots as inverse power reconstruction

A root is not primitive extraction of a fractional occurrence. It is the inverse reconstruction problem for a previously defined finite power map.

Definition 51.18 (Exact constructive root)**.** *Let N be a nonnegative finite count report and let $n > 0$ be a depth report. A nonnegative report Q is an exact constructive nth root of N when*

$$\mathrm{Pow}_n(Q) =_{\mathrm{count}} N.$$

If the full source structure is also required to match, one must strengthen this to a declared ledger correspondence

$$\mathrm{Pow}_n(Q) \cong N.$$

The distinction matters. Equal counts can certify a scalar root without proving that the radicand was actually constructed by the candidate root ledger.

Definition 51.19 (Perfect power)**.** *A nonnegative count report N is a perfect nth power when an exact finite root report exists at depth n.*

For example, the scalar report 16 is both a perfect square and a perfect fourth power:

$$16 = 4^2 = 2^4.$$

These are different inverse reconstructions of the same projected count.

51.9 Monotonicity and uniqueness of nonnegative roots

Lemma 51.20 (Strict monotonicity of positive-depth natural powers)**.** *Let $0 \le a < b$ and let $n > 0$. Then*

$$a^n < b^n.$$

Proof. A strict containment of finite carriers gives an injection from the a-carrier into the b-carrier together with at least one unmatched occurrence in the latter. Applying the injection at every depth position embeds every a-tuple into a b-tuple. Since $n > 0$, a tuple using an unmatched b-occurrence at one retained position lies outside the image. Hence the product containment is strict. □

Theorem 51.21 (Uniqueness of the nonnegative exact root)**.** *For fixed positive depth n, a nonnegative finite count has at most one nonnegative exact scalar nth root.*

Proof. If $q_1 < q_2$, strict monotonicity gives $q_1^n < q_2^n$, so the two powers cannot both equal the same count. □

This theorem establishes uniqueness only for the nonnegative scalar root report. Different source ledgers can still share the same root count.

51.10 Finite root shells and retained residuals

Most finite counts are not perfect powers at a chosen depth. The constructive inverse state must retain the nearest complete power below the radicand and the unresolved residual above that shell.

Definition 51.22 (Lower root-shell index)**.** *For a nonnegative finite count N and positive depth n, the lower root-shell index is the unique nonnegative count q satisfying*

$$q^n \le N < (q+1)^n.$$

Existence follows because the finite sequence

$$0^n, 1^n, 2^n, \ldots, N^n$$

contains a first power exceeding N unless N is already bounded at the preceding stage; strict monotonicity gives uniqueness.

Definition 51.23 (Root-shell residual)**.** *Given the lower root-shell index q, define*

$$\rho_n(N) = N - q^n.$$

The complete finite root state is

$$\mathrm{RootState}_n(N) = (N, n, q, \rho_n(N)),$$

with

$$0 \le \rho_n(N) < (q+1)^n - q^n.$$

The upper bound is the width of the next power shell. The state is exact precisely when

$$\rho_n(N) = 0_{N:q^n}.$$

Theorem 51.24 (Exact-root closure)**.** *A finite nonnegative count N has exact nth root q if and only if its root-shell residual is the typed zero*

$$\rho_n(N) = 0.$$

The residual is not an error to be silently discarded. It is the positive source difference between the radicand and the largest completed $n$fold self-product below it.

Example 51.25 (Non-exact square state)**.** *For $N = 10$ and depth two,*

$$3^2 = 9 \le 10 < 16 = 4^2.$$

The finite root state is

$$\mathrm{RootState}_2(10) = (10, 2, 3, 1).$$

The report 3 is the lower square-shell index, not an exact square root of 10. Any later decimal or real report for $\sqrt{10}$ must refine this retained shell relation rather than pretending the residual was already zero.

51.11 Signed roots

Signed root reconstruction depends on depth parity.

Theorem 51.26 (Finite signed-root conditions)**.** *Let $n > 0$.*

(i) A positive exact radicand has one nonnegative exact root. When n is even and the root magnitude is nonzero, it also has an oppositely oriented root with the same scalar power.

(ii) A negative exact radicand can have an integer root only when n is odd; that root is negatively oriented.

(iii) A negative radicand has no even-depth integer root.

(iv) The zero radicand has the zero root for every positive depth.

Proof. The parity theorem for signed powers shows that every even-depth power is nonnegative, while odd-depth power preserves the base orientation. Nonnegative magnitude uniqueness then fixes the root magnitude whenever an exact root exists. □

For a positive radicand and even depth, the two signed solutions

$$q \qquad \text{and} \qquad -q$$

are source-distinct orientations with equal scalar power. A *principal root* is obtained only after a convention selects the nonnegative orientation. The radical symbol alone should not be allowed to hide that convention.

51.12 Exact roots of ratios

Let $\mathcal{R}(A, B)$ have positive denominator B.

Definition 51.27 (Exact rational root)**.** *A ratio $\mathcal{R}(C, D)$ is an exact nth root of $\mathcal{R}(A, B)$ when*

$$\mathcal{R}(\mathrm{Pow}_n(C), \mathrm{Pow}_n(D)) =_{\mathrm{ratio}} \mathcal{R}(A, B).$$

A sufficient source-faithful condition is that numerator and denominator admit exact roots:

$$A \cong \mathrm{Pow}_n(C), \qquad B \cong \mathrm{Pow}_n(D).$$

But ratio equality can also permit common-carrier reductions, so exact scalar rational-root status must be checked after declared normalization.

Not every positive rational ratio has a rational exact root. For example, no finite rational carrier has square equal to the ratio 2/1. Constructing such a root requires a later completion grammar. The present section records only the finite shell and exact-ratio predicates; it does not manufacture an irrational object prematurely.

51.13 Power words are not automatically geometric forms

The words *square* and *cube* are often used for both operational depth and geometric enclosure. Those meanings must be separated.

Definition 51.28 (Operational square)**.** *The operational square of A is the two-depth product ledger*

$$\mathrm{Pow}_2(A) = A \times A.$$

It contains ordered pairs of A-occurrences.

Definition 51.29 (Operational cube)**.** *The operational cube of A is the three-depth product ledger*

$$\mathrm{Pow}_3(A) = (A \times A) \times A,$$

up to declared reassociation.

Neither construction alone supplies geometric enclosure. A geometric square additionally requires four boundary placements, their incidence order, closure, and whatever equal-length and angle conditions the chosen grammar declares. A geometric cube requires a three-dimensional boundary structure and cannot be inferred from a scalar third power alone.

Therefore

$$A^2$$

means repeated product depth unless a geometric interpretation has been separately constructed. It does not prove that A has been arranged as a Euclidean square.

Likewise, taking a scalar square root of an area report does not automatically recover a primitive line object. It recovers a side magnitude only under a declared square-area grammar. The minimum triangular area closure developed earlier is not constructively generated by squaring one line report.

Principle 51.30 (Operational depth before geometric naming)**.** *Powers count repeated operational composition. Geometric squares, cubes, and higher-dimensional cells require additional placement, incidence, closure, and measure structure.*

51.14 Power units and typed operations

When a base report carries a declared unit or object type, repeated product changes the type of the report.

If A has unit type u, then the depth-n product carries the recursively constructed product type

$$u^{\otimes n}.$$

Only after a unit algebra has been declared may this be abbreviated as u^n.

An exact root operation must therefore invert not only a scalar count but also the declared product type. A numerical equality such as

$$q^n = N$$

does not prove that a typed quantity Q is a valid root of a typed radicand unless the unit and construction grammars also close.

This prevents a scalar coincidence from being mistaken for dimensional or semantic identity.

51.15 Operational-depth preservation theorem

Theorem 51.31 (Power-ledger preservation)**.** *A scalar power report preserves its finite constructive source only when the reporting map retains, or proves irrelevant, all of the following:*

(i) the base carrier and its type;

(ii) the depth carrier and whether its orientation is positive, zero, or negative;

(iii) the ordered factor roles and tuple addresses;

(iv) the grouping and reassociation rule;

(v) the signed orientation and parity route;

(vi) any reciprocal construction used for negative depth;

(vii) the zero-depth identity convention;

(viii) and every residual or typed zero relevant to an inverse root state.

Equal scalar powers do not by themselves imply equal power ledgers.

Theorem 51.32 (Root-state preservation)**.** *A reported root value preserves its finite inverse construction only when it retains, or proves irrelevant:*

(i) the radicand source;

(ii) the power depth;

(iii) the selected orientation or principal-root convention;

(iv) the candidate root carrier;

(v) the exact power comparison;

(vi) the lower and upper power shells when the root is not exact;

(vii) and the root-shell residual.

A decimal approximation without this state is a later projection, not the primitive root construction.

The finite dependency now extends as

$$\begin{aligned}\text{multiplication} &\longrightarrow \text{repeated self-product} \longrightarrow \text{power depth}\\ &\longrightarrow \text{signed and reciprocal powers} \longrightarrow \text{inverse power constraint}\\ &\longrightarrow \text{exact root or retained root-shell residual.}\end{aligned}$$

The next section must distinguish a finite recursive rule from a completed infinite object. Only after that distinction is explicit can recurring expansions, unbounded power families, sequences, limits, and completion be introduced without treating an ellipsis as though it were an already constructed infinity.

Part X
Relations, Mappings, and Indexed Construction Families

52 Relations, mappings, and indexed construction families

The preceding sections have already used source-preserving correspondences, projections, transpositions, reassociations, extraction rules, and recurrence steps. Those constructions now permit a general account of relation and function. A function must not be inserted as a primitive black box. It is a declared rule relating already distinguishable source occurrences to already distinguishable target occurrences.

The present section remains finite. It constructs relations, mappings, composition, inverse relations, fibers, and indexed families without yet asserting a completed infinite domain or an infinite sequence.

52.1 A relation begins with retained source and target roles

Let A and B be constructed reports. Their placement product

$$A \times B$$

contains ordered pairs (a, b) with a retained source role from A and a retained target role from B.

Definition 52.1 (Binary relation report)**.** *A binary relation from A to B is a source-preserving subreport*

$$\mathcal{R} \preceq_{\mathrm{C}} A \times B.$$

An occurrence $(a, b) \in \mathcal{R}$ reports that the source occurrence a is related to the target occurrence b under the declared relation.

The order of the pair is part of the relation ledger. In general,

$$(a, b) \neq_{\mathrm{C}} (b, a).$$

A symmetric scalar statement about the pair does not erase its source and target roles.

Definition 52.2 (Relation membership)**.** *For a relation report $\mathcal{R} \preceq_{\mathrm{C}} A \times B$, write*

$$a \,\mathcal{R}\, b$$

when the ordered occurrence (a, b) is admitted to $\mathcal{R}$.

The notation $a \,\mathcal{R}\, b$ is therefore a projection of the retained pair occurrence. It does not create the relation.

52.2 Converse, diagonal, and relational identity

Definition 52.3 (Converse relation)**.** *The converse of $\mathcal{R} \preceq_{\mathrm{C}} A \times B$ is*

$$\mathcal{R}^{-1} = \{(b, a) : (a, b) \in \mathcal{R}\} \preceq_{\mathrm{C}} B \times A.$$

At this stage the superscript -1 means reversal of relation orientation. It does not yet assert that the converse is a function.

Definition 52.4 (Diagonal relation)**.** *The diagonal relation on A is*

$$\Delta_A = \{(a, a) : a \in A\} \preceq_{\mathrm{C}} A \times A.$$

The diagonal retains two roles for the same admitted occurrence: source role and target role. Its scalar values may agree, but the relation still records an input placement and an output placement.

Definition 52.5 (Relational composition)**.** *Let*

$$\mathcal{R} \preceq_{\mathrm{C}} A \times B, \qquad \mathcal{S} \preceq_{\mathrm{C}} B \times C.$$

Their composition is the relation

$$\mathcal{S} \circ \mathcal{R} = \{(a, c) : \textit{there is a retained } b \in B \textit{ with } (a, b) \in \mathcal{R} \textit{ and } (b, c) \in \mathcal{S}\}.$$

The intermediate b is part of the full composition ledger. Projecting only (a, c) discards the route through B.

Theorem 52.6 (Relational identity)**.** *For every relation $\mathcal{R} \preceq_{\mathrm{C}} A \times B$,*

$$\mathcal{R} \circ \Delta_A = \mathcal{R}, \qquad \Delta_B \circ \mathcal{R} = \mathcal{R}$$

at the admitted-pair level.

Proof. The diagonal adds no new target choice. Composing on the source side retains exactly the pairs (a, b) already in $\mathcal{R}$, because the only diagonal intermediate paired with a is a itself. The target-side argument is analogous. □

52.3 Input support and output support

A relation need not use every occurrence of either carrier.

Definition 52.7 (Source support)**.** *The source support of $\mathcal{R} \preceq_{\mathrm{C}} A \times B$ is*

$$\mathrm{src}(\mathcal{R}) = \{a \in A : \textit{some } b \in B \textit{ satisfies } (a, b) \in \mathcal{R}\}.$$

Definition 52.8 (Target support)**.** *The target support of $\mathcal{R}$ is*

$$\mathrm{tgt}(\mathcal{R}) = \{b \in B : \textit{some } a \in A \textit{ satisfies } (a, b) \in \mathcal{R}\}.$$

These supports are later reported as domain and image when the relation satisfies the function condition. The ambient target carrier B remains distinct from the target occurrences actually reached.

52.4 Single-output construction

A general relation permits one source to relate to no target, one target, or several targets. A deterministic mapping requires a declared single-output condition.

Definition 52.9 (Single-valued relation)**.** *A relation $\mathcal{R} \preceq_{\mathrm{C}} A \times B$ is single-valued on a source report $D \preceq_{\mathrm{C}} A$ when, for every $a \in D$, any two retained pairs*

$$(a, b_1) \in \mathcal{R}, \qquad (a, b_2) \in \mathcal{R}$$

have the same target occurrence under the declared target identity:

$$b_1 =_{\mathrm{C}} b_2.$$

Equal projected target values are insufficient when target occurrence identity matters. If

$$\pi(b_1) = \pi(b_2)$$

but $b_1 \neq_{\mathrm{C}} b_2$, then the relation has two source-distinct target routes unless a later equivalence deliberately identifies them.

Definition 52.10 (Totality on a declared source)**.** *A relation $\mathcal{R} \preceq_{\mathrm{C}} A \times B$ is total on $D \preceq_{\mathrm{C}} A$ when every $a \in D$ occurs as the source of at least one admitted pair in $\mathcal{R}$.*

Definition 52.11 (Constructive mapping)**.** *A constructive mapping*

$$f : D \longrightarrow B$$

is a relation report $\Gamma_f \preceq_{\mathrm{C}} D \times B$ that is total and single-valued on D, together with any rule, trace, or source structure declared relevant to producing its target occurrences.

The pair report Γ_f is the graph projection of the mapping. The complete constructive mapping may contain more than the graph.

52.5 Rule, graph, and evaluation ledger

Three objects must be kept distinct.

Definition 52.12 (Mapping rule)**.** *A mapping rule ρ_f is a declared construction that, when supplied an admissible input occurrence, selects or constructs a target occurrence.*

Definition 52.13 (Extensional graph)**.** *The extensional graph of f is the pair report*

$$\Gamma_f = \{(a, f(a)) : a \in D\} \preceq_{\mathrm{C}} D \times B.$$

Definition 52.14 (Evaluation ledger)**.** *An evaluation ledger for input a is a retained construction*

$$\Lambda_f(a) = (a, \rho_f, \tau_a, r_a, f(a)),$$

where τ_a denotes any admitted transition route and r_a denotes any admitted residual, quotient, boundary, or closure state produced during evaluation.

The graph reports input-output pairs. The evaluation ledger reports how the output was constructed. Two mapping rules may have the same graph while retaining different evaluation ledgers.

Principle 52.15 (Graph equality is a projection)**.** *The implication*

$$\Gamma_f = \Gamma_g \Longrightarrow f =_{\mathrm{C}} g$$

is valid only if the declared identity criterion for mappings discards every rule, route, residual, and construction difference between f and g.

This does not reject extensional function equality. It identifies it as a declared quotient of richer constructive mappings.

52.6 Domain, codomain, and image

Definition 52.16 (Declared domain)**.** *The declared domain of*

$$f : D \to B$$

is the source report D on which the graph is total and single-valued.

Definition 52.17 (Declared codomain)**.** *The declared codomain is the target carrier B into which outputs are admitted.*

Definition 52.18 (Image)**.** *The image of f is the reached target support*

$$\mathrm{im}(f) = \{b \in B : \text{some } a \in D \text{ satisfies } f(a) =_{\mathrm{C}} b\}.$$

The codomain and image need not be equal. A codomain is part of the mapping declaration; the image is constructed by actual output occurrences.

Definition 52.19 (Partial mapping)**.** *A partial mapping from A to B is a single-valued relation whose source support is a proper subreport*

$$D \preceq_{\mathrm{C}} A.$$

It may be written

$$f : A \rightharpoonup B,$$

with constructed domain $D = \mathrm{src}(\Gamma_f)$.

Undefinedness is not automatically a target value. If the mapping has no admitted output for a, then either a lies outside the constructed domain or a separate typed failure report must be constructed in an enlarged codomain.

52.7 Injective, surjective, and bijective mappings

Definition 52.20 (Injective mapping)**.** *A mapping $f : D \to B$ is injective when*

$$f(a_1) =_{\mathrm{C}} f(a_2) \Longrightarrow a_1 =_{\mathrm{C}} a_2.$$

Equivalently, every reached target occurrence has at most one source occurrence in the graph.

Definition 52.21 (Surjective mapping)**.** *A mapping $f : D \to B$ is surjective onto its declared codomain when*

$$\mathrm{im}(f) =_{\mathrm{C}} B.$$

Definition 52.22 (Bijective mapping)**.** *A mapping is bijective when it is both injective and surjective.*

These properties depend on the chosen identity criteria for source and target occurrences. A projection may collapse several target occurrences to one scalar value and thereby destroy injectivity even when the pre-projection mapping was injective.

Example 52.23 (Projection destroys injectivity)**.** *Let*

$$D = \{(0, 1), (1, 0)\}$$

and let f be the identity mapping on these ordered cells. Then f is injective. Under the scalar projection

$$\pi_+(i, j) = i + j,$$

both outputs report 1. The composed report $\pi_+ \circ f$ is not injective on D. The failure belongs to the projection fiber, not to the original identity mapping.

52.8 Composition of mappings

Let

$$f : A \to B, \qquad g : B \to C.$$

Definition 52.24 (Constructive composition)**.** *The composite mapping*

$$g \circ f : A \to C$$

has graph

$$\Gamma_{g \circ f} = \{(a, c) : \text{the retained intermediate } b = f(a) \text{ satisfies } c = g(b)\}.$$

Its complete evaluation ledger retains

$$\Lambda_{g \circ f}(a) = (a, \Lambda_f(a), f(a), \Lambda_g(f(a)), g(f(a))).$$

Theorem 52.25 (Mapping composition)**.** *If $f : A \to B$ and $g : B \to C$ are total single-valued mappings, then $g \circ f : A \to C$ is total and single-valued.*

Proof. Each $a \in A$ has one target $b = f(a)$, and that b has one target $c = g(b)$. Therefore each a has one composite target c. □

At graph level, mapping composition is associative. At ledger level, the two parenthesizations

$$h \circ (g \circ f) \qquad \text{and} \qquad (h \circ g) \circ f$$

retain canonically corresponding nested evaluation structures. They need not be literally identical syntactic ledgers unless reassociation is included in the declared equality.

52.9 Identity mapping

Definition 52.26 (Identity mapping)**.** *The identity mapping on A is the mapping*

$$\mathrm{id}_A : A \to A$$

with graph

$$\Gamma_{\mathrm{id}_A} = \Delta_A.$$

For every $a \in A$,

$$\mathrm{id}_A(a) = a.$$

The identity output is the same admitted occurrence under the declared identity criterion. The evaluation ledger may still record that an identity operation was invoked.

Theorem 52.27 (Mapping identity laws)**.** *For mappings $f : A \to B$,*

$$\mathrm{id}_B \circ f = f, \qquad f \circ \mathrm{id}_A = f$$

at graph level. Ledger equality additionally requires the declared normalization of inserted identity steps.

52.10 Inverse relation and inverse mapping

The converse relation Γ_f^{-1} always exists as a reversed pair report. It is not always a mapping.

Theorem 52.28 (Inverse-function criterion)**.** *Let $f : A \to B$ be a mapping. Its converse graph defines a mapping*

$$f^{-1} : B \to A$$

exactly when f is bijective onto B.

Proof. The converse is single-valued precisely when no two source occurrences of f reach the same target, which is injectivity. It is total on B precisely when every target in B is reached, which is surjectivity. □

When f is bijective,

$$f^{-1} \circ f = \mathrm{id}_A, \qquad f \circ f^{-1} = \mathrm{id}_B$$

at graph level.

A noninjective mapping has inverse fibers rather than a single-valued inverse. A nonsurjective mapping has target occurrences with no converse source. These are structural conditions, not failures of notation.

52.11 Projections and fibers

Definition 52.29 (Reporting projection)**.** *A reporting projection is a mapping*

$$\pi : C \to V$$

from a constructive carrier C to a selected report carrier V.

A projection may preserve all relevant structure, or it may deliberately discard source, order, orientation, residual, route, width, base, or closure information.

Definition 52.30 (Fiber)**.** *For $v \in V$, the fiber of π over v is*

$$\pi^{-1}(v) = \{c \in C : \pi(c) = v\}.$$

Two constructive objects are projection-equal under π when they lie in the same fiber:

$$c_1 =_\pi c_2 \quad \Longleftrightarrow \quad \pi(c_1) = \pi(c_2).$$

Theorem 52.31 (Injective projection criterion)**.** *Projection equality implies constructive identity on a declared subreport $D \preceq_{\mathrm{C}} C$ exactly when the restricted projection*

$$\pi|_D : D \to V$$

is injective under the declared identities.

The theorem does not say that noninjective projections are invalid. It says that an additional proof is required before an output report may stand in for the full source object.

52.12 Higher-arity relations and mappings

An n-input relation is constructed on a product carrier

$$A_1 \times A_2 \times \cdots \times A_n \times B.$$

A mapping of n inputs is a total single-valued relation from the input product

$$A_1 \times \cdots \times A_n$$

to B.

Thus a binary operation on A is a mapping

$$\star : A \times A \to A.$$

Closure is not supplied by the symbol $\star$. It is the property that every admitted input pair has an admitted output in A.

The arithmetic operations constructed earlier can now be retrospectively encoded as mappings on their declared carriers. This encoding does not replace their source constructions.

52.13 Indexed construction families

A family is not initially an infinite set. It is a mapping from a constructed index report into a target carrier.

Definition 52.32 (Finite indexed family)**.** *Let I be a finite index report and let X be a target carrier. A finite indexed family in X is a mapping*

$$x : I \to X.$$

The occurrence at index i is written

$$x_i = x(i).$$

The full family report retains at least

$$(I, X, \Gamma_x),$$

and may additionally retain the rule and evaluation ledger producing each x_i.

Definition 52.33 (Finite sequence report)**.** *A finite sequence is an indexed family whose index carrier carries the constructed successor order*

$$i_0 \to i_1 \to \cdots \to i_m.$$

Order is therefore supplied by the index carrier and its successor relation. The target values need not themselves be ordered.

Definition 52.34 (Reindexing of a family)**.** *Given a bijection*

$$\sigma : J \to I,$$

the reindexed family is

$$x \circ \sigma : J \to X.$$

A reindexing may preserve the multiset of output reports while changing adjacency, order, recurrence, or causal route. Therefore a sequence property is reindexing-invariant only after that invariance is proved.

52.14 Transition maps and finite recurrences

Definition 52.35 (Transition map)**.** *A transition map on a state carrier X is a mapping*

$$T : X \to X.$$

Given an initial state x_0, finite iteration constructs

$$x_1 = T(x_0), \qquad x_2 = T(x_1), \qquad \ldots, \qquad x_m = T(x_{m-1}).$$

For a constructed finite index carrier

$$I_m = [0, 1, \ldots, m],$$

the recurrence ledger is the finite indexed family

$$x^{(m)} : I_m \to X, \qquad x^{(m)}(k) = T^k(x_0),$$

where T^k abbreviates the already constructed finite composition depth.

Definition 52.36 (Recurrence state)**.** *A recurrence state at index k is the complete ledger*

$$\Lambda_T(k) = (x_0, k, T, x_k, \tau_k),$$

where τ_k retains the sequence of intermediate transitions when that route is relevant.

Equal endpoint states need not imply equal recurrence histories. A cycle may return to a prior state after a positive transition depth:

$$x_j =_{\mathrm{C}} x_k \qquad \text{with} \qquad j < k.$$

The endpoint equality does not erase the completed transition count $k - j$.

52.15 Finite family equality

Several equality criteria are possible.

Definition 52.37 (Pointwise report equality)**.** *Families $x : I \to X$ and $y : I \to X$ are pointwise equal under a report projection π when*

$$\pi(x(i)) = \pi(y(i))$$

for every $i \in I$.

Definition 52.38 (Constructive family equality)**.** *The families are constructively equal when the index carrier, index roles, target occurrences, mapping rule, and every other declared relevant ledger component are preserved by the chosen correspondence.*

Pointwise scalar equality may leave different source routes at every index. Conversely, two differently indexed families may be isomorphic under a declared reindexing while not being literally the same report.

52.16 Mapping-preservation theorem

Theorem 52.39 (Constructive mapping preservation)**.** *A report of a mapping preserves the complete finite mapping only when it retains, or proves irrelevant, all of the following:*

(i) the source carrier and its identity criterion;

(ii) the target carrier and its identity criterion;

(iii) the admitted domain;

(iv) the input-output graph;

(v) the rule selecting or constructing each output;

(vi) every intermediate state, route, quotient, residual, boundary, and closure component relevant to the property under study;

(vii) the equality relation imposed on rules and evaluation ledgers;

(viii) and any normalization that deletes identity steps, source tags, ordering, or repeated equal outputs.

Equality of output graphs alone proves complete mapping identity only after all omitted components have been shown irrelevant under the declared identity criterion.

Theorem 52.40 (Indexed-family preservation)**.** *A finite indexed-family report preserves its construction only when it retains, or proves irrelevant:*

(i) the index carrier;

(ii) the successor, order, or adjacency relation on indices;

(iii) the target carrier;

(iv) the mapping from indices to target occurrences;

(v) the generating rule, when one is used;

(vi) the initial and terminal conditions;

(vii) and the evaluation route or recurrence ledger relevant to the stated property.

A displayed list of values is therefore not automatically the complete family object.

The dependency now extends as

$$\begin{aligned}\text{ordered product} &\longrightarrow \text{relation report} \longrightarrow \text{single-output condition} \longrightarrow \text{mapping}\\ &\longrightarrow \text{composition and inverse criteria} \longrightarrow \text{projection fibers}\\ &\longrightarrow \text{finite indexed families and recurrence ledgers.}\end{aligned}$$

The next section must distinguish a finite rule that admits another iteration from a completed infinite carrier. A finite recurrence description, an ellipsis, and an asserted infinity are not yet the same constructive object.

Part XI
Finite Rules and Indefinite Continuation

53 Finite rules, extensible prefixes, and continuation

The constructions developed so far are finite. A numeral has finitely many retained digit positions, a quotient–remainder computation has finitely many completed extraction stages, a power has finite operational depth, and a sequence report has a finite index carrier. Mathematics nevertheless uses notations such as

$$x_0, x_1, x_2, \dots$$

and phrases such as "continue the construction indefinitely." These expressions must not silently replace a finite construction by an already completed infinite carrier.

This section distinguishes four objects that conventional notation often compresses into one:

(i) a completed finite prefix;

(ii) a finite rule that constructs a further stage;

(iii) a proof that the rule remains applicable after every completed finite stage;

(iv) and a completed infinite carrier, should such a carrier later be admitted.

Only the first three are constructed here. The fourth requires a separate completion or admission rule and an identity criterion not yet introduced.

53.1 Finite prefixes

Definition 53.1 (Finite prefix report)**.** *Let X be a state carrier and let*

$$I_m = [0, 1, \dots, m]$$

be a constructed finite successor-index carrier. A finite prefix report in X is a finite indexed family

$$x^{[m]} : I_m \to X.$$

Its displayed form is

$$[x_0, x_1, \ldots, x_m], \qquad x_k = x^{[m]}(k).$$

The superscript $[m]$ records the terminal index of the completed prefix. It is not an exponent. The prefix contains exactly the occurrences attached to the indices already admitted into I_m.

Definition 53.2 (Prefix restriction)**.** *For $j \le m$, the restriction of $x^{[m]}$ to I_j is*

$$x^{[m]} \restriction_{I_j} : I_j \to X.$$

A prefix $x^{[j]}$ is compatible with $x^{[m]}$ when

$$x^{[m]} \restriction_{I_j} =_{\mathrm{C}} x^{[j]}$$

under the declared family identity.

Compatibility says that the longer report extends the earlier report without revising its retained occurrences, indices, or relevant construction ledger.

Definition 53.3 (One-stage extension)**.** *A one-stage extension of $x^{[m]}$ is a prefix*

$$x^{[m+1]} : I_{m+1} \to X$$

such that

$$x^{[m+1]} \restriction_{I_m} =_{\mathrm{C}} x^{[m]}.$$

The newly admitted occurrence is

$$x_{m+1} = x^{[m+1]}(m+1).$$

An extension therefore preserves the old prefix and adds one whole indexed occurrence. It does not retroactively turn the earlier finite prefix into an incomplete fragment of an object that had already existed in completed form.

53.2 Continuation rules

Definition 53.4 (Continuation rule)**.** *A continuation rule on a class of admissible finite prefixes is a mapping or relation*

$$E : x^{[m]} \longmapsto x^{[m+1]}$$

that constructs at least one compatible one-stage extension whenever its input prefix satisfies the rule's declared admissibility conditions.

A continuation rule may use only the terminal state, the entire retained prefix, an auxiliary state ledger, or some combination of them. These are different rule types and must not be identified without proof.

Definition 53.5 (State-local continuation)**.** *A continuation rule is state-local when there exists a transition map*

$$T : X \to X$$

such that

$$x_{m+1} = T(x_m).$$

Definition 53.6 (History-dependent continuation)**.** *A continuation rule is history-dependent when the next occurrence is selected from the complete prefix ledger:*

$$x_{m+1} = F(x^{[m]}).$$

Equal terminal states guarantee equal next states only for a state-local deterministic rule. A history-dependent rule may assign different continuations to prefixes sharing the same terminal value.

Definition 53.7 (Continuation certificate)**.** *A continuation certificate for a prefix $x^{[m]}$ is a retained report*

$$\kappa_m$$

showing that the admissibility conditions of E are satisfied at stage m and recording the construction of at least one compatible extension.

The complete one-stage continuation state is therefore not merely x_{m+1}. It may be represented as

$$\Lambda_E(m) = (x^{[m]}, E, \kappa_m, x^{[m+1]}).$$

53.3 Deterministic and branching continuation

Definition 53.8 (Deterministic continuation)**.** *A continuation rule is deterministic on its admissible prefixes when every admissible prefix has exactly one compatible next prefix under the declared identity:*

$$E(x^{[m]}) =_{\mathrm{C}} x^{[m+1]}.$$

Definition 53.9 (Branching continuation)**.** *A continuation relation is branching when some admissible prefix has more than one constructively distinct compatible extension:*

$$x_a^{[m+1]} \neq_{\mathrm{C}} x_b^{[m+1]}.$$

A displayed common prefix does not identify a unique indefinite continuation in the branching case. The branch-selection rule or branch address is part of the report.

Example 53.10 (Ambiguous continuation)**.** *The finite list*

$$[1, 2, 3]$$

can be extended by many rules, including

$$[1, 2, 3, 4], \qquad [1, 2, 3, 5], \qquad [1, 2, 3, 6],$$

or a rule that returns to 1*. The finite prefix alone does not determine which rule was intended.*

Thus the notation

$$1, 2, 3, \ldots$$

is not self-interpreting. Its meaning depends on a separately declared continuation grammar.

53.4 The ellipsis is a report token

Definition 53.11 (Ellipsis token)**.** *An ellipsis token is a distinguishable report object indicating that a declared continuation rule is intended to remain available beyond the displayed finite prefix.*

The mark ... is not the unrendered collection of every omitted occurrence. It occupies one report position and refers to a rule or continuation obligation.

Accordingly,

$$[x_0, x_1, x_2, \ldots]$$

contains, as written, finitely many displayed occurrences and one continuation token. Its interpretation requires at least:

(i) the displayed prefix;

(ii) the continuation rule;

(iii) the stage at which the rule begins;

(iv) any state or history needed by the rule;

(v) and the identity criterion for later extensions.

Theorem 53.12 (Ellipsis noncompletion)**.** *An ellipsis token does not by itself construct a completed infinite carrier.*

Proof. The token is one finite report occurrence. It may refer to an extension rule, but reference to a rule does not supply all of the rule's possible future output occurrences as already admitted indexed members. A completed carrier would additionally require an admission rule for all its indices and an identity criterion for the resulting total report. Neither is supplied by the mark alone. □

53.5 Indefinite extendability

Definition 53.13 (Indefinitely extendable construction)**.** *A prefix construction governed by E is indefinitely extendable when, for every completed finite admissible prefix $x^{[m]}$, a continuation certificate constructs at least one compatible extension $x^{[m+1]}$.*

In retrospective numerical notation, the condition is

$$\text{for every constructed finite } m, \quad x^{[m]} \text{ admits an } E\text{-extension } x^{[m+1]}.$$

This is a statement about the availability of a next finite construction at every finite stage. It is not yet the assertion that one completed object containing all stages has been constructed.

Theorem 53.14 (No terminal prefix)**.** *If a continuation rule is indefinitely extendable, no admissible finite prefix is terminal under that rule.*

Proof. Let $x^{[m]}$ be any admissible completed finite prefix. Indefinite extendability supplies a continuation certificate and a compatible extension $x^{[m+1]}$. Hence $x^{[m]}$ is not terminal. Since the argument applies to every completed finite admissible prefix, none is terminal. □

Corollary 53.15 (No last constructed natural stage)**.** *For every constructed finite stage index m, the successor stage $m+1$ can be constructed. Hence there is no greatest finite natural-number report under the successor construction.*

This corollary establishes unbounded finite succession. It does not yet identify the unbounded succession rule with a completed set of all natural numbers.

53.6 Can continue, is continued, and has been completed

Three claims must be distinguished:

availability: $x^{[m]}$ admits another stage;
execution: $x^{[m+1]}$ has actually been constructed;
completion: a total carrier containing every stage has been admitted.

The first does not imply that the next stage has been executed. The second produces only a longer finite prefix. Repeating the first two claims at every finite stage still does not, without another construction principle, supply the third claim.

Principle 53.16 (Nonretroactive extension)**.** *The construction of a later prefix does not alter the finite identity of an earlier prefix. It adds a new compatible report containing the earlier one as a retained restriction.*

Thus

$$x^{[m]} \neq_{\mathrm{C}} x^{[m+1]}$$

even though

$$x^{[m+1]} \restriction_{I_m} =_{\mathrm{C}} x^{[m]}.$$

53.7 Universal finite-stage statements

A theorem may apply uniformly to every finite stage without first treating all stages as members of a completed carrier.

Definition 53.17 (Uniform finite-stage rule)**.** *A finite-stage property P is uniform under E when:*

(i) $P(x^{[0]})$ *is established for the initial prefix;*

(ii) whenever $P(x^{[m]})$ *holds and* E *constructs* $x^{[m+1]}$*, the property* $P(x^{[m+1]})$ *follows by one retained extension argument.*

This is the constructive content later compressed into induction. The induction report records an initial construction and a successor-preservation rule. It does not require that every stage have already been separately rendered.

Theorem 53.18 (Finite-stage induction)**.** *If a property has a constructed initial witness and is preserved by every admissible one-stage extension, then the property holds for each finite prefix obtained by finitely many applications of the extension rule.*

Proof. The initial prefix carries the property. Each completed extension carries the property from its predecessor to its successor. Therefore any prefix reached after a specified finite number of extension steps inherits the property through a finite chain of preserved reports. □

The theorem concerns every finitely reached stage. It does not yet define a property of a completed infinite totality.

53.8 Coherent prefix systems

Definition 53.19 (Coherent prefix system)**.** *A coherent prefix system is a continuation rule together with a family of finite prefix reports satisfying*

$$x^{[m+1]} \restriction_{I_m} =_{\mathrm{C}} x^{[m]}$$

at every constructed stage.

A coherent system preserves all prior finite information under extension. It is the finite data needed for a later completion construction, but it is not itself automatically identical to the completed object that such a construction may introduce.

Definition 53.20 (Prefix projection)**.** *For $j \leq m$, define the prefix projection*

$$\rho_{m,j}(x^{[m]}) = x^{[m]} \restriction_{I_j} .$$

Coherence requires

$$\rho_{m,j}(x^{[m]}) =_{\mathrm{C}} x^{[j]}.$$

For $i \leq j \leq m$, restriction composes:

$$\rho_{m,i} = \rho_{j,i} \circ \rho_{m,j}.$$

This equation reports compatibility among finite observations. A later completed carrier, if admitted, must reproduce every finite prefix under its own finite-report projections.

53.9 Potential continuation and completed infinity

Definition 53.21 (Potential continuation report)**.** *A potential continuation report is the finite package*

$$\mathcal{P}_E = (x^{[0]}, E, \mathcal{K}),$$

where $x^{[0]}$ is an initial prefix, E is a continuation rule, and $\mathcal{K}$ is a uniform certificate scheme establishing extension after every completed finite admissible stage.

The word *potential* here does not mean imaginary or uncertain. It means that each further stage is constructible when requested, while no terminal totality has yet been admitted.

Definition 53.22 (Completed infinite carrier – deferred)**.** *A completed infinite carrier would be a report object X_∞ equipped with an index-admission rule and finite projections such that every constructed finite prefix is recovered compatibly:*

$$\rho_m(X_\infty) =_{\mathrm{C}} x^{[m]}.$$

This definition states the obligation such an object would have to satisfy, but this section does not assert its existence. Existence requires a later completion principle and a declared equality relation among candidate completions.

Theorem 53.23 (Quantifier separation)**.** *The statement*

$$\textit{for every finite } m\textit{, an admissible prefix } x^{[m]} \textit{ can be constructed}$$

does not by itself construct an object X_∞ containing every stage as an already admitted member.

Proof. The first statement supplies a uniform finite construction scheme and, for any selected finite m, a finite witness. The second statement introduces a new total report object, a total index carrier, and compatibility maps from that object to every finite prefix. Those additional objects and relations are not components of any one finite witness and do not follow merely by renaming the extension scheme. □

The theorem does not deny that a completed carrier may be consistently constructed. It requires the construction to be stated rather than smuggled in through notation.

53.10 The infinity symbol is not an infinite carrier

The mark

$$\infty$$

is a distinguishable finite report token. Depending on the declared grammar, it may report unboundedness, a direction of continuation, a projective boundary, divergence, or a completed extension object. These meanings are not interchangeable.

Principle 53.24 (Typed infinity)**.** *Every use of ∞ must declare the carrier and relation in which it is interpreted. The existence of the symbol does not supply infinitely many indexed occurrences, and two uses of the symbol need not denote the same construction.*

For example,

$$m \longmapsto m + 1$$

reports unbounded finite succession, whereas

$$\lim_{m \to \infty} x_m$$

will later report a limiting relation over that succession. Neither expression has yet constructed a scalar object that may be manipulated as an ordinary finite count.

53.11 Finite recurrence and cyclic compression

A nonterminating recurrence need not require an infinite amount of descriptive information.

Suppose a deterministic transition map

$$T : S \to S$$

acts on a finite state carrier S. If no terminal state is reached, the recurrence eventually revisits a prior state. Once

$$x_j =_{\mathrm{C}} x_k, \qquad j < k,$$

the subsequent state sequence repeats with cycle length $k - j$ under the same deterministic rule.

Definition 53.25 (Finite recurrence descriptor)**.** *A finite recurrence descriptor is the retained package*

$$\mathcal{C} = (x_0, T, j, k, [x_0, \ldots, x_k]),$$

where $x_j =_{\mathrm{C}} x_k$ certifies the entrance and closure of a repeated state cycle.

The descriptor finitely determines each later finite stage. It does not consist of infinitely many already written state occurrences.

Theorem 53.26 (Cycle compression)**.** *For a deterministic recurrence on a finite state carrier, a finite transient prefix together with one retained cycle determines the state at every requested finite stage.*

Proof. Before the cycle entrance, the retained transient prefix supplies the state directly. After the entrance, divide the additional transition count by the positive cycle length. The remainder selects the corresponding retained position in the cycle. This uses the quotient–remainder construction already established and produces the requested finite state without constructing every intervening occurrence as a permanent total carrier. □

53.12 Recurring radix reports

The decimal report

$$0.333\ldots_{10}$$

should now be read as a finite continuation report. For the ratio $1/3$, the radix recurrence retains the state

$$10 \cdot 1 = 3 \cdot 3 + 1.$$

The remainder returns to 1, so the same digit and remainder state recur.

The exact finite descriptor may be written

$$\mathcal{C}_{1/3,10} = (R_0 = 1,\ d_1 = 3,\ R_1 = 1,\ E_{10,3}).$$

This descriptor constructs any requested finite decimal prefix

$$0.3, \quad 0.33, \quad 0.333, \quad \ldots$$

while retaining the unresolved remainder at each finite stage.

Theorem 53.27 (Recurring report exactness)**.** *A recurring radix descriptor may exactly specify a rational ratio even though no finite digit prefix is itself equal to the completed positional value later associated with the recurrence.*

Proof. The descriptor retains the original numerator, denominator, base, digit output, and repeated remainder state. These finite data determine the quotient–remainder relation exactly. Each finite written prefix, by contrast, terminates at a finite place and retains a nonzero residual obligation. Exactness belongs to the recurrence relation, not to any one truncated digit list. □

This distinction will be essential for statements such as

$$0.999\ldots = 1.$$

At present, the left side is a finite recurrence descriptor and each of its finite prefixes differs from the unit. Equality with a completed real unit cannot be stated until a completion relation is constructed.

53.13 Indefinite sums and products are deferred

The notation

$$\sum_{k=0}^{\infty} a_k$$

or

$$\prod_{k=0}^{\infty} a_k$$

does not yet denote a completed sum or product. What is currently constructed is the finite family of partial operations

$$S_m = \sum_{k=0}^{m} a_k, \qquad P_m = \prod_{k=0}^{m} a_k,$$

for each requested finite m.

The continuation rules

$$S_{m+1} = S_m + a_{m+1}, \qquad P_{m+1} = P_m \cdot a_{m+1}$$

construct longer finite ledgers. A completed infinite aggregate requires a later convergence or completion criterion.

Principle 53.28 (No aggregation by punctuation)**.** *An infinity mark or ellipsis does not perform infinitely many additions, multiplications, unions, intersections, or evaluations. It reports a continuation grammar whose completed aggregate, if any, must be separately constructed.*

53.14 Stage-local and completion-dependent properties

Definition 53.29 (Stage-local property)**.** *A property is stage-local when its truth for a continuation report can be decided from a specified finite prefix and its retained ledger.*

Examples include the value at a selected finite index, whether a selected transition occurred, and whether a finite remainder closed at a specified stage.

Definition 53.30 (Completion-dependent property)**.** *A property is completion-dependent when no specified finite prefix alone supplies its declared witness and the property instead refers to the behavior of all finite extensions or to a completed carrier.*

Convergence, equality of completed recurring expansions, boundedness over every stage, and the value of an infinite series are completion-dependent. They cannot be inferred from a few observed prefixes unless a uniform theorem supplies the missing universal relation.

Theorem 53.31 (Finite observation insufficiency)**.** *No finite observed prefix, by itself, proves that an unspecified continuation rule is indefinitely extendable, terminating, recurring, convergent, or unique.*

Proof. Any finite prefix is compatible with multiple later rules unless the continuation grammar has been separately fixed. Some compatible rules terminate, some continue, some branch, and some repeat. Therefore the observed values alone do not determine a global continuation property. □

53.15 Continuation-preservation theorem

Theorem 53.32 (Continuation-report preservation)**.** *A finite report of an indefinitely extendable construction preserves the construction only when it retains, or proves irrelevant, all of the following:*

(i) the initial prefix and its complete identity;

(ii) the finite index and successor grammar;

(iii) the continuation rule;

(iv) the admissibility conditions for each extension;

(v) the continuation certificates or a uniform certificate scheme;

(vi) whether continuation is deterministic or branching;

(vii) every branch-selection address relevant to the displayed route;

(viii) the compatibility relation among successive prefixes;

(ix) any terminal, recurrence, or closure conditions;

(x) and the distinction between an extension scheme and a completed carrier.

An ellipsis or infinity symbol without these declarations is not a lossless report of the construction.

Theorem 53.33 (Finite-rule noncollapse)**.** *Let r map a continuation ledger to a displayed prefix with an ellipsis. If two distinct continuation rules produce the same displayed prefix, then r is noninjective. Any property depending on which continuation occurs cannot factor through the displayed ellipsis report without an additional fiber-constancy theorem.*

Proof. Let C_1 and C_2 share the same displayed prefix but use different continuation rules. Then

$$r(C_1) = r(C_2)$$

while

$$C_1 \neq_{\mathrm{C}} C_2.$$

A property that distinguishes their later stages is not constant on the fiber of r. Hence the common displayed report cannot determine that property. □

53.16 Dependency summary

The construction developed in this section is

$$\begin{aligned}\text{finite indexed prefix} &\longrightarrow \text{compatible one-stage extension} \longrightarrow \text{continuation rule}\\ &\longrightarrow \text{uniform extension certificate} \longrightarrow \text{indefinite extendability}\\ &\longrightarrow \text{coherent finite-prefix system.}\end{aligned}$$

It does not yet make the further step

$$\text{coherent finite-prefix system} \longrightarrow \text{completed infinite carrier.}$$

That step requires a declared completion construction, a criterion for equality among completed reports, and a proof that every finite projection is preserved.

The next section should unify the equality relations used throughout the manuscript and establish the exact fiber condition under which a later projection may replace a prior constructive ledger. Only after that audit should convergence and completion identify distinct finite approximation routes with one completed value.

Part XII
Equality Hierarchy and Projection-Fiber Preservation

54 Equality hierarchy and projection-fiber preservation

The manuscript has now constructed several ways in which two reports may agree. These agreements must not be compressed into one untyped equality symbol. An equality statement is always relative to the structure that the comparison is required to preserve.

A report may agree with another in count while differing in source, agree in projected value while differing in orientation, agree pointwise while differing in evaluation history, or agree under a completion that has not yet been constructed. Every substitution therefore requires a declared equality relation and a theorem showing that the property being substituted is preserved by that relation.

54.1 Equality is a typed comparison relation

Definition 54.1 (Typed equality)**.** *Let $\mathcal{C}$ be a carrier of constructive reports and let Σ be a declared family of structures to be preserved. The relation*

$$C_1 =_\Sigma C_2$$

means that C_1 and C_2 agree under the identity criterion determined by Σ.

The declaration Σ may include any combination of:

(i) indexed occurrence identity;

(ii) source labels;

(iii) report order;

(iv) orientation;

(v) grouping or bracketing;

(vi) transition history;

(vii) boundary and closure obligations;

(viii) quotient–remainder state;

(ix) base and positional width;

(x) evaluation trace;

(xi) or only a selected scalar or coordinate report.

The bare symbol $=$ is therefore safe only after the relevant carrier and preservation criterion have already been fixed. Where those conditions are not evident, the equality must be typed explicitly.

Principle 54.2 (No untyped substitution)**.** *From $C_1 =_\Sigma C_2$, one may substitute C_1 for C_2 only in a context proved invariant under $=_\Sigma$.*

This principle is the equality analogue of the earlier warning that equal projected outputs do not imply equal constructions.

54.2 Constructive identity

Definition 54.3 (Constructive identity)**.** *Two reports are constructively identical, written*

$$C_1 =_{\mathrm{C}} C_2,$$

when every structure declared constitutive of the reports is preserved, including every relevant indexed occurrence, source address, relation, orientation, boundary, and construction obligation.

Constructive identity is the strongest equality used in this manuscript. It does not mean that two marks printed at different locations are literally the same physical ink event. It means that, within the declared mathematical report, no retained constitutive distinction separates them.

Thus

$$C_1 =_{\mathrm{C}} C_2 \quad \Longrightarrow \quad \pi(C_1) = \pi(C_2)$$

for every projection π defined on the retained structure, but the converse generally fails.

Theorem 54.4 (Constructive identity preserves every declared property)**.** *Let P be a property defined entirely from the structures included in constructive identity. If*

$$C_1 =_{\mathrm{C}} C_2,$$

then

$$P(C_1) = P(C_2).$$

Proof. By definition, constructive identity preserves every structure from which P is evaluated. Replacing one report by the other therefore changes no input to the evaluation of P. □

This theorem is strong because the equality relation is strong. Most later report equalities deliberately forget some structure and therefore preserve fewer properties.

54.3 Occurrence, source, and order equality

Definition 54.5 (Occurrence equality)**.** *Two report occurrences*

$$\omega_i = (i, v_i), \qquad \omega_j = (j, v_j)$$

are occurrence-identical only when their complete occurrence addresses agree under the declared report identity.

Value equality

$$v_i = v_j$$

does not imply occurrence equality when $i \neq j$.

Definition 54.6 (Source equality)**.** *Two reports are source-equal when their retained source addresses correspond and every source-dependent relation is preserved.*

Definition 54.7 (Order equality)**.** *Two finite reports are order-equal when corresponding occurrences appear under the same declared successor and precedence relations.*

A permutation may preserve multiplicity and projected values while failing order equality. A source-erasing projection may preserve order while failing source equality. These are independent comparison obligations.

54.4 Structural equivalence and isomorphism

Definition 54.8 (Structure-preserving equivalence)**.** *Let $\mathcal{S}(C)$ denote a selected structural signature of a report C. Write*

$$C_1 \cong_{\mathcal{S}} C_2$$

when there exists a bijective correspondence between their admitted components that preserves every relation in $\mathcal{S}$.

For example,

$$(A \times B) \times C \cong A \times (B \times C)$$

under reassociation, although their nested source syntax differs. Likewise,

$$A \times B \cong B \times A$$

under transposition at the product-structure level, although ordered product orientations differ.

Theorem 54.9 (Isomorphism preserves structural invariants)**.** *If*

$$C_1 \cong_{\mathcal{S}} C_2,$$

then every property defined solely from the preserved signature $\mathcal{S}$ has equal reports on C_1 and C_2.

An isomorphism therefore licenses substitution only for $\mathcal{S}$-invariant properties. It does not prove source identity, construction-history identity, or equality of structures omitted from $\mathcal{S}$.

54.5 Reindexing and extensional equality

Definition 54.10 (Reindexing equality)**.** *Two indexed reports are reindexing-equal when a bijection between their index carriers preserves the admitted occurrences while changing only the names or presentation order of the indices declared irrelevant.*

Definition 54.11 (Extensional value equality)**.** *Two reports are extensionally value-equal when they produce the same collection of projected values after occurrence multiplicity, order, source, and any other omitted structure have been quotiented away.*

Thus

$$[a_0, a_1] \quad \text{with} \quad \pi(a_0) = \pi(a_1) = a$$

may project extensionally to

$$\{a\},$$

as does the one-occurrence report $[a_0]$. The extensional reports agree while constructive multiplicity does not.

Principle 54.12 (Extensional equality is domain-limited)**.** *Extensional equality is sufficient only for properties that depend solely on the retained extensional value collection.*

Counting source occurrences, reconstructing routes, distinguishing repeated evaluations, and auditing cancellation cannot in general be performed from extensional value equality alone.

54.6 Count, scalar, coordinate, and ratio equality

Definition 54.13 (Count equality)**.** *Two finite reports are count-equal when their occurrence carriers admit a bijection:*

$$C_1 =_{\text{count}} C_2 \quad \Longleftrightarrow \quad |I_{C_1}| = |I_{C_2}|.$$

Count equality ignores the values, sources, order, and relations occupying those indices unless separately retained.

Definition 54.14 (Scalar-report equality)**.** *Let*

$$\pi_{\text{sc}} : \mathcal{C} \to S$$

be a scalar reporting map. Then

$$C_1 =_{\text{sc}} C_2 \quad \Longleftrightarrow \quad \pi_{\text{sc}}(C_1) = \pi_{\text{sc}}(C_2).$$

Definition 54.15 (Coordinate equality)**.** *Let*

$$\pi_{\text{coord}} : \mathcal{C} \to K$$

be a declared coordinate projection. Coordinate equality means equality in K after that projection.

Coordinate equality does not imply equality of the pre-coordinate placements or relations unless the coordinate projection is injective over the relevant domain.

Definition 54.16 (Ratio equality)**.** *For positive denominator reports, constructive ratios $\mathcal{R}(A, B)$ and $\mathcal{R}(C, D)$ are ratio-equal when their cross-product counts balance:*

$$|A||D| = |C||B|.$$

Ratio equality preserves the normalized comparison while generally discarding numerator and denominator source identities. Thus

$$\frac{1}{2} =_{\text{ratio}} \frac{2}{4}$$

without requiring the two ratio ledgers to be constructively identical.

54.7 Orientation, balance, and closure equality

Definition 54.17 (Orientation equality)**.** *Two signed reports are orientation-equal when their positive and negative lanes agree after the declared cancellation and normalization rules.*

A positive lane and a negative lane may have equal magnitude while differing in orientation. Magnitude equality alone does not imply orientation equality.

Definition 54.18 (Balance equality)**.** *Two balance reports are balance-equal when they have the same retained carrier and the same signed imbalance state.*

In particular,

$$H(A; B) = 0$$

reports equal complementary counts relative to the retained carrier B. Another zero balance produced by a different carrier or decomposition may be scalar-equal to zero while remaining a different balance ledger.

Definition 54.19 (Closure equality)**.** *Two operation states are closure-equal when the same declared obligations have been discharged and no relevant unmatched source, inverse, boundary, or remainder survives.*

Closure equality is stronger than equality of a selected net scalar. A scalar zero can result from opposing nonzero contributions, while a closure zero additionally certifies that every declared obligation has closed.

54.8 Pointwise, rule, and family equality

Definition 54.20 (Pointwise equality)**.** *Mappings $f, g : X \to Y$ are pointwise equal on X when*

$$f(x) = g(x)$$

for every admitted $x \in X$.

Pointwise equality compares output reports. It need not preserve implementation, evaluation trace, resource ledger, or source route.

Definition 54.21 (Rule equality)**.** *Two mapping or continuation rules are rule-equal when their admissible inputs, transition conditions, branch conditions, outputs, and retained evaluation obligations agree under the declared rule grammar.*

Definition 54.22 (Indexed-family equality)**.** *Two indexed families are equal under a declared family criterion only when both their index correspondence and their member comparison satisfy that criterion.*

Two families may be pointwise scalar-equal while differing in source, recurrence rule, or proof certificate. Therefore

$$f(x) = g(x) \text{ for all } x$$

does not by itself imply

$$f =_{\mathrm{C}} g$$

as constructive mappings.

54.9 Report-induced equivalence relations

Let

$$r : \mathcal{C} \to V$$

be any reporting map.

Definition 54.23 (Report-induced equivalence)**.** *Define*

$$C_1 \sim_r C_2 \quad \Longleftrightarrow \quad r(C_1) = r(C_2).$$

The relation $\sim_r$ is an equivalence relation on $\mathcal{C}$ whenever equality in V is an equivalence relation.

Proof. Reflexivity follows from $r(C) = r(C)$. Symmetry follows from symmetry of equality in V. Transitivity follows because $r(C_1) = r(C_2)$ and $r(C_2) = r(C_3)$ imply $r(C_1) = r(C_3)$. □

Definition 54.24 (Projection fiber)**.** *For $v \in \mathrm{im}(r)$, the fiber of r over v is*

$$r^{-1}(v) = \{C \in \mathcal{C} : r(C) = v\}.$$

Equivalently, the fiber of C is its equivalence class

$$[C]_r = \{D \in \mathcal{C} : D \sim_r C\}.$$

A projection fiber is exactly the family of constructive reports that the selected report map treats as equal.

Example 54.25 (Addition fiber)**.** *For the grid report*

$$\pi_+(i, j) = i + j,$$

the scalar fiber over 1 *contains at least*

$$\pi_+^{-1}(1) = \{(0, 1), (1, 0)\}.$$

The scalar value 1 *identifies the fiber but does not identify which oriented source pair produced it.*

54.10 Quotients are deliberate information contracts

The quotient carrier

$$\mathcal{C}/\!\sim_r$$

contains one equivalence class for each report value in $\mathrm{im}(r)$. Passing from $\mathcal{C}$ to this quotient deliberately declares every distinction within each fiber irrelevant for the quotient-level task.

Principle 54.26 (Quotient declaration)**.** *A quotient does not prove that the forgotten distinctions never existed. It declares that they will not be distinguished by the quotient identity criterion.*

The quotient may be mathematically useful and exactly suited to a chosen task. The remaining question is whether a later property is well defined on that quotient. That requires a preservation theorem.

54.11 Projection-fiber factorization theorem

Let

$$P : \mathcal{C} \to Z$$

be any property, measurement, predicate, or further report of the full constructive ledger.

Theorem 54.27 (Projection-fiber factorization)**.** *Let $r : \mathcal{C} \to V$ and $P : \mathcal{C} \to Z$. The following statements are equivalent:*

(i) There exists a map

$$\widehat{P} : \mathrm{im}(r) \to Z$$

such that

$$P = \widehat{P} \circ r.$$

(ii) P is constant on every fiber of r; that is, for all $C_1, C_2 \in \mathcal{C}$,

$$r(C_1) = r(C_2) \quad \Longrightarrow \quad P(C_1) = P(C_2).$$

Proof. Assume first that $P = \widehat{P} \circ r$. If $r(C_1) = r(C_2)$, then

$$P(C_1) = \widehat{P}(r(C_1)) = \widehat{P}(r(C_2)) = P(C_2).$$

Hence P is constant on every fiber.

Conversely, suppose that P is constant on every fiber of r. For each $v \in \mathrm{im}(r)$, select any $C \in \mathcal{C}$ satisfying $r(C) = v$ and define

$$\widehat{P}(v) = P(C).$$

If D is another representative with $r(D) = v$, then C and D lie in the same fiber, so $P(C) = P(D)$. Therefore $\widehat{P}$ is well defined. For every $C \in \mathcal{C}$,

$$\widehat{P}(r(C)) = P(C),$$

and hence $P = \widehat{P} \circ r$. □

Corollary 54.28 (Exact condition for ledger erasure)**.** *A constructive ledger may be discarded with respect to a property P exactly when P is constant on every fiber of the retained report map.*

This condition is necessary and sufficient. It is not a convention, intuition, or appeal to the usefulness of scalar notation.

54.12 Predicate preservation

A predicate is a map

$$Z : \mathcal{C} \to \{\mathrm{true}, \mathrm{false}\}.$$

Corollary 54.29 (Predicate descent)**.** *A predicate Z is well defined on the quotient induced by r if and only if*

$$r(C_1) = r(C_2) \quad \Longrightarrow \quad Z(C_1) = Z(C_2).$$

Thus a projected equality licenses a projected predicate only when every constructive report in the same fiber receives the same predicate value.

Example 54.30 (Scalar-zero fiber)**.** *Let*

$$\pi_{\mathrm{sc}} : \mathcal{C} \to S$$

be a scalar projection, and let Z_{cl} report complete closure. Scalar zero determines complete closure only if

$$\pi_{\mathrm{sc}}(C_1) = \pi_{\mathrm{sc}}(C_2) = 0 \quad \Longrightarrow \quad Z_{\mathrm{cl}}(C_1) = Z_{\mathrm{cl}}(C_2)$$

for every admissible pair in the scalar-zero fiber.

If one zero-output ledger is completely closed while another retains an unmatched source or boundary obligation, then complete closure is not constant on the zero fiber. In that case scalar zero cannot represent complete operator zero.

54.13 Injectivity is sufficient but not necessary

Theorem 54.31 (Injective-report sufficiency)**.** *If $r : \mathcal{C} \to V$ is injective, then every property $P : \mathcal{C} \to Z$ factors through r on* $\mathrm{im}(r)$*.*

Proof. Every fiber of an injective map contains exactly one constructive report. Every property is therefore constant on each fiber, and the factorization theorem applies. □

Injectivity is stronger than necessary. A noninjective projection may still preserve a particular property if that property is constant on the larger fibers. For example, a count projection may discard object labels while still preserving whether the count is even. It cannot thereby preserve every property of the original report.

Principle 54.32 (Property-by-property preservation)**.** *Noninjective report maps require a separate fiber-constancy proof for every later property that is to be inferred from the report alone.*

54.14 Counterexample to universal projection preservation

Theorem 54.33 (Universal history erasure is false)**.** *There is no general theorem stating that equal reported outputs preserve every property of their constructive ledgers.*

Proof. Let $C_1 \neq_{\mathrm{C}} C_2$ satisfy

$$r(C_1) = r(C_2).$$

Define the property

$$P(C) = \begin{cases} 0, & C = C_1, \\ 1, & C \neq C_1. \end{cases}$$

Then

$$P(C_1) \neq P(C_2)$$

while their reports agree. Hence P is not constant on the common fiber and cannot factor through r. □

The theorem does not say that no history may ever be erased. It says that erasure is legitimate only relative to properties proved insensitive to the erased distinctions.

54.15 Congruence and lawful substitution

Let $\sim$ be an equivalence relation on a carrier $\mathcal{C}$ and let

$$F : \mathcal{C}^n \to \mathcal{D}$$

be an n-input operation.

Definition 54.34 (Congruence for an operation)**.** *The relation $\sim$ is a congruence for F when*

$$C_k \sim D_k \quad \textit{for every } k$$

implies

$$F(C_1, \ldots, C_n) \sim' F(D_1, \ldots, D_n)$$

under the declared output equivalence $\sim'$.

Theorem 54.35 (Quotient operation criterion)**.** *An operation F descends to a well-defined operation on equivalence classes if and only if its output class is constant under replacement of each input by any equivalent representative.*

Proof. If the quotient operation is well defined, equivalent representatives must produce the same output class. Conversely, if the output class is invariant under representative replacement, define the quotient operation by selecting any representatives; the result is independent of the selection. □

This is the multi-input version of fiber factorization. It supplies the exact requirement behind symbolic substitution rules.

Example 54.36 (Ratio representatives)**.** *The pairs $(1, 2)$ and $(2, 4)$ represent the same ratio class. Addition and multiplication of ratio classes are well defined because the cross-product equivalence is a congruence for the constructed ratio operations. Source-specific properties of the original numerator and denominator reports do not descend merely because the ratio operations do.*

54.16 Canonical representatives do not restore erased history

A quotient system may select one canonical representative from each equivalence class. For example, a ratio may be reduced, a signed ledger may be normalized, or a positional numeral may drop leading zero digits.

Definition 54.37 (Canonicalization)**.** *A canonicalization map is a function*

$$c : \mathcal{C} \to \mathcal{C}$$

that assigns a chosen representative to each report equivalence class and satisfies

$$r(c(C)) = r(C).$$

Principle 54.38 (Canonical representative limitation)**.** *A canonical representative records the chosen normal form of a fiber. It does not reconstruct which original member of the fiber produced the report.*

If

$$C_1 \sim_r C_2$$

and

$$c(C_1) = c(C_2),$$

then canonicalization has intentionally merged the source distinction. Applying c twice may stabilize the normal form,

$$c(c(C)) = c(C),$$

but this idempotence is not recovery of the original ledger.

54.17 Sections and representative choice

A right inverse or section of r is a map

$$s : \mathrm{im}(r) \to \mathcal{C}$$

satisfying

$$r(s(v)) = v.$$

Such a section chooses one constructive representative for each report value.

Theorem 54.39 (Representative selection is not inversion)**.** *If r is noninjective, no section s can satisfy*

$$s(r(C)) =_{\mathrm{C}} C$$

for every $C \in \mathcal{C}$.

Proof. Let $C_1 \neq_{\mathrm{C}} C_2$ lie in the same fiber, so $r(C_1) = r(C_2) = v$. The section returns the single report $s(v)$. It cannot be constructively identical to both distinct reports. Therefore at least one original ledger is not recovered. □

A decoding rule can invert a projection only after the relevant domain has been restricted so that the projection is injective, or after sufficient side information has been retained to separate the fibers.

54.18 Minimal sufficient reports

Definition 54.40 (Sufficient report for a property family)**.** *Let $\mathcal{P}$ be a family of properties on $\mathcal{C}$. A report map $r : \mathcal{C} \to V$ is sufficient for $\mathcal{P}$ when every $P \in \mathcal{P}$ factors through r.*

Definition 54.41 (Property indistinguishability)**.** *For a property family $\mathcal{P}$, define*

$$C_1 \sim_{\mathcal{P}} C_2 \quad \Longleftrightarrow \quad P(C_1) = P(C_2) \text{ for every } P \in \mathcal{P}.$$

A report is sufficient for $\mathcal{P}$ precisely when every report fiber lies inside one $\mathcal{P}$-indistinguishability class.

Theorem 54.42 (Sufficiency criterion)**.** *The report map r is sufficient for $\mathcal{P}$ if and only if*

$$r(C_1) = r(C_2) \quad \Longrightarrow \quad C_1 \sim_{\mathcal{P}} C_2.$$

This theorem converts the vague question "Does the report contain enough information?" into an exact comparison between report fibers and the distinctions made by the intended property family.

54.19 Composition of projections

Suppose

$$\mathcal{C} \xrightarrow{r} V \xrightarrow{q} W$$

are successive report maps. The composite report is

$$q \circ r : \mathcal{C} \to W.$$

Every fiber of r lies inside a fiber of $q \circ r$, but the composite may merge several r-fibers.

Theorem 54.43 (Successive-erasure criterion)**.** *A property $P : \mathcal{C} \to Z$ factors through $q \circ r$ if and only if P is constant on every fiber of the composite report. Factoring through r alone does not guarantee factoring through the coarser report $q \circ r$.*

Thus preservation must be re-audited after every additional projection. A count may preserve parity; reducing parity further to a constant symbol does not.

54.20 Equality ladders

The equality relations developed so far form a preservation ladder rather than a single chain valid in every case. A typical sequence is

$$\text{constructive identity} \Longrightarrow \text{source-preserving structural equivalence} \Longrightarrow \text{count or scalar equality},$$

but reverse implications require additional injectivity or fiber-constancy theorems.

For ratio reports one may have

$$\mathcal{R}(A,B) =_{\mathrm{C}} \mathcal{R}(C,D) \Longrightarrow \mathcal{R}(A,B) =_{\mathrm{ratio}} \mathcal{R}(C,D),$$

while

$$\mathcal{R}(A,B) =_{\mathrm{ratio}} \mathcal{R}(C,D) \not\Longrightarrow \mathcal{R}(A,B) =_{\mathrm{C}} \mathcal{R}(C,D).$$

For signed reports one may have

$$C_1 =_{\mathrm{C}} C_2 \Longrightarrow \mathrm{net}(C_1) = \mathrm{net}(C_2),$$

while equal net values need not preserve the positive and negative source lanes.

For mapping evaluations,

$$E_1 =_{\mathrm{C}} E_2 \Longrightarrow \mathrm{out}(E_1) = \mathrm{out}(E_2),$$

but equal outputs do not imply equal evaluation traces.

Principle 54.44 (Direction of implication)**.** *Every weakening of equality may be used in the forward direction established by its projection. No reverse implication is available without a separate recovery or preservation theorem.*

54.21 Zero-predicate hierarchy

The manuscript now distinguishes several zero reports:

(i) a zero occurrence count;

(ii) a typed no-contribution report;

(iii) an exact subtraction closure;

(iv) an exact division remainder zero;

(v) a signed balance zero;

(vi) a scalar or coordinate zero;

(vii) and a complete closure zero in which every declared obligation has closed.

These zero reports may share the scalar symbol 0 while answering different questions.

Definition 54.45 (Zero-predicate projection)**.** *Let Z_{full} be a complete zero predicate on a ledger space $\mathcal{C}$, and let*

$$\pi_0 : \mathcal{C} \to S$$

be a selected scalar report. Scalar zero represents complete zero exactly when there exists a scalar predicate $\widehat{Z}$ such that

$$Z_{\mathrm{full}} = \widehat{Z} \circ \pi_0.$$

By the factorization theorem, this is equivalent to complete zero being constant on every scalar fiber, especially the scalar-zero fiber.

Theorem 54.46 (No automatic promotion of zero)**.** *The equation*

$$\pi_0(C) = 0$$

may be promoted to

$$Z_{\text{full}}(C) = \text{true}$$

only after proving that every admissible ledger with scalar report zero satisfies the complete zero predicate.

This theorem will later govern the relation between complex scalar zero and complete operator closure.

54.22 Critical-half preservation condition

The constructive half was derived from the integer balance lane

$$H(A; B) = 2|A| - |B|.$$

The normalized scalar report is

$$\nu(A; B) = \frac{|A|}{|B|}.$$

On the retained complement domain,

$$H(A; B) = 0 \quad \Longrightarrow \quad \nu(A; B) = \frac{1}{2}.$$

The reverse implication is valid only when the scalar half report is tied to the same retained carrier and complement grammar. A bare occurrence of the scalar value $1/2$ does not reconstruct which carrier, partition, orientation, or reflection relation produced it.

Theorem 54.47 (Critical-half fiber condition)**.** *Let ν be a normalized coordinate report and let Z_H be the predicate $H = 0$. The coordinate statement $\nu = 1/2$ determines constructive balance exactly when Z_H is constant on the $1/2$ fiber of ν over the admitted domain.*

This is not yet an RH claim. It is the preservation condition that any later critical-half application must satisfy.

54.23 Finite-rule equality and future completion

Section 53 distinguished a finite continuation descriptor from a completed infinite carrier. The present equality hierarchy sharpens that distinction.

Two continuation descriptors may:

(i) share every displayed finite prefix encountered so far;

(ii) produce equal values at every requested finite index;

(iii) be pointwise equal on all finite indices;

(iv) use different source rules or proof certificates;

(v) or later be identified by a completion relation not yet constructed.

Pointwise equality of all finite values is stronger than agreement on a sampled prefix, but it still need not imply constructive rule identity. A later completion may deliberately quotient multiple approximation families into one completed value. That quotient will be legitimate only for properties constant on its completion fibers.

Principle 54.48 (Completion must declare its equality)**.** *A completion is not merely the addition of new objects. It must declare which finite approximation families are identified, prove that the relation is an equivalence relation, and verify that every operation and property used on the completion is well defined on those equivalence classes.*

This is the equality work that must precede any assertion such as

$$0.999\ldots = 1$$

or any identification of distinct convergent sequences with one real report.

54.24 Equality-audit protocol

For every equality used later in the manuscript, the following audit should be possible:

(i) What are the two compared report carriers?

(ii) Which structures are required to be preserved?

(iii) Which structures are intentionally forgotten?

(iv) What map or correspondence induces the equality?

(v) What are the fibers or equivalence classes?

(vi) Is the later property constant on those fibers?

(vii) Does the relevant operation respect representative replacement?

(viii) Is a canonical representative being confused with an original source ledger?

(ix) Is a scalar zero being promoted to a stronger closure zero?

(x) Is a later completion equality being retrojected into a finite stage?

An equality claim that cannot answer these questions is not yet sufficiently typed for Primitive Constructive Logic.

54.25 Equality-preservation theorem

Theorem 54.49 (General equality-preservation theorem)**.** *Let $\sim$ be an equivalence relation on a constructive carrier $\mathcal{C}$, and let*

$$q : \mathcal{C} \to \mathcal{C}/\!\sim$$

be the quotient report. A property, predicate, or operation is well defined using only the quotient report if and only if it is invariant under replacement by $\sim$-equivalent representatives.

Proof. For a one-input property this is the projection-fiber factorization theorem applied to q. For a multi-input operation, apply the same argument to the product quotient and require output equivalence to be constant on every product fiber. □

Corollary 54.50 (No free reverse equality)**.** *From a weaker report equality one may infer a stronger constructive equality only when the quotient fibers have been proved singleton on the relevant domain or when additional retained data select the original member uniquely.*

54.26 Dependency summary

The hierarchy developed in this section is

$$\begin{aligned}\text{constructive reports} &\longrightarrow \text{declared comparison structure} \longrightarrow \text{typed equality}\\ &\longrightarrow \text{report-induced equivalence classes} \longrightarrow \text{projection fibers}\\ &\longrightarrow \text{fiber-constancy test} \longrightarrow \text{lawful quotient property or operation.}\end{aligned}$$

The central theorem is

$$\boxed{P = \widehat{P} \circ r \quad \Longleftrightarrow \quad r(C_1) = r(C_2) \Rightarrow P(C_1) = P(C_2).}$$

Consequently:

(1) equal reports need not be equal constructions;

(2) a quotient is a declared information contract, not proof that erased distinctions are unreal;

(3) injectivity preserves every property, but noninjective projections preserve only fiber-constant properties;

(4) symbolic substitution requires congruence;

(5) canonicalization does not reconstruct source history;

(6) scalar zero does not automatically imply complete closure;

(7) the scalar half does not automatically reconstruct its carrier and balance lane;

(8) and every future completion must prove that its equality and operations are well defined on completion fibers.

The next section should construct convergence from signed residuals, nested whole bounds, and finite-resolution agreement. It must distinguish exact finite equality from limit equality and must use the present fiber theorem to state precisely what a completed value preserves and what it discards.

Part XIII
Limits and Completed Number Systems

55 Constructive convergence and limits

A limit cannot be used to explain the arithmetic, signed orientation, order, ratio, power, mapping, continuation, or equality structures on which its own definition depends. Those structures have now

been constructed. The present section can therefore define convergence without treating a positive primitive as though it literally became zero and without presupposing a completed real-number carrier.

The central distinction is

$$\boxed{\text{finite-stage residual} \neq_{\mathrm{C}} \text{limit-value report.}}$$

A residual may remain positive at every finite stage while the indexed family of residuals satisfies every requested positive resolution. Convergence is a relation among a finite continuation rule, its finite stages, a candidate target, and a resolution grammar. It is not the physical or logical disappearance of any fixed positive occurrence.

55.1 The ordered approximation carrier

The signed natural, integer, ratio, and reciprocal constructions developed earlier provide a carrier of finite signed ratio reports. Denote the resulting ordered carrier by

$$\mathbb{Q}_{\mathrm{C}}.$$

The symbol names the already constructed family of finite signed ratios; it does not assume a geometric number line or a completed continuum.

For $X, Y \in \mathbb{Q}_{\mathrm{C}}$, define the signed difference report

$$\Delta(X, Y) := X \oplus (-Y),$$

where addition and orientation reversal are the retained signed operations constructed in Section 50 and common-denominator ratio arithmetic is retained from Sections 39–41.

Definition 55.1 (Residual magnitude)**.** *The* residual magnitude *between X and Y is the nonnegative ratio report*

$$d_{\mathrm{C}}(X, Y) := \|\Delta(X, Y)\|_{\mathrm{C}},$$

obtained by forgetting only the surviving orientation of the normalized signed difference while retaining its positive magnitude carrier.

The notation d_{C} is introduced as a report of finite signed difference. No topological or real-valued distance has been assumed.

Lemma 55.2 (Finite residual laws)**.** *For $X, Y, Z \in \mathbb{Q}_{\mathrm{C}}$:*

(i) $d_{\mathrm{C}}(X, Y) \geq 0$*;*

(ii) $d_{\mathrm{C}}(X, Y) = 0$ *exactly when X and Y are equal as normalized signed-ratio reports;*

(iii) $d_{\mathrm{C}}(X, Y) = d_{\mathrm{C}}(Y, X)$*;*

(iv)

$$d_{\mathrm{C}}(X, Z) \leq d_{\mathrm{C}}(X, Y) + d_{\mathrm{C}}(Y, Z).$$

Proof. The first three statements follow from signed normal form and orientation reversal. For the fourth, place the three finite ratios over a common positive denominator. The direct residual from X to Z is obtained by joining the residual from X to Y with the residual from Y to Z and then allowing additional opposing cancellations. Cancellation cannot increase the surviving magnitude, so the direct magnitude is bounded by the joined magnitudes. □

Definition 55.3 (Positive resolution report)**.** *A* positive resolution report *is a finite ratio* $\rho \in \mathbb{Q}_{\mathrm{C}}$ *satisfying*

$$0 < \rho.$$

It specifies a requested positive residual bound. It is not an infinitesimal and is not identified with zero.

Every positive finite ratio admits a strictly smaller positive resolution. For example,

$$0 < \frac{\rho}{2} < \rho.$$

Thus the approximation grammar can be refined without declaring any fixed positive object to be zero.

55.2 Approximation families and residual ledgers

Definition 55.4 (Finite-stage approximation family)**.** *An* approximation family *in* $\mathbb{Q}_{\mathrm{C}}$ *is an indexed construction family*

$$X = (X_n)_{n \in \mathbb{N}}$$

generated by a retained finite rule, recurrence, explicit lookup report, or other admitted mapping

$$f_X : \mathbb{N} \longrightarrow \mathbb{Q}_{\mathrm{C}}.$$

Each X_n *is a completed finite ratio report.*

The family descriptor is finite when its rule and required source data are finite, even though the rule admits another finite stage for every natural index. Section 53 already established that this indefinite extendability does not itself construct a completed infinite carrier.

Definition 55.5 (Target residual ledger)**.** *Let* $L \in \mathbb{Q}_{\mathrm{C}}$ *be a candidate target. The* target residual ledger *of* X *relative to* L *is*

$$E_n(X; L) := \Delta(X_n, L),$$

with magnitude

$$e_n(X; L) := d_{\mathrm{C}}(X_n, L).$$

The full ledger retains the orientation, magnitude, source route, and finite index of every residual.

A positive residual orientation reports that X_n lies above L in the signed order; a negative residual orientation reports that it lies below; and a zero residual reports exact finite-stage equality. The magnitude report alone forgets the side of approach.

Definition 55.6 (Tail)**.** *For a natural index* N*, the* tail beginning at N *is the restricted family*

$$X_{\geq N} := (X_n)_{n \geq N}.$$

This is a continuation-rule restriction, not an already completed infinite set.

55.3 Finite-resolution containment

Definition 55.7 (Resolution neighborhood)**.** *For $L \in \mathbb{Q}_{\mathrm{C}}$ and positive resolution ρ, define the finite-resolution neighborhood report*

$$B_\rho(L) := \{X \in \mathbb{Q}_{\mathrm{C}} : d_{\mathrm{C}}(X, L) < \rho\}.$$

Membership means that the finite signed residual between X and L is constructively smaller than the declared positive resolution.

The set notation is retrospective compression of the membership rule. It does not posit a spatial ball or a completed continuum.

Definition 55.8 (Eventual resolution containment)**.** *The family X is* eventually contained at resolution ρ around L *when there exists a finite index N_ρ such that*

$$n \geq N_\rho \quad \Longrightarrow \quad X_n \in B_\rho(L).$$

Equivalently,

$$n \geq N_\rho \quad \Longrightarrow \quad e_n(X; L) < \rho.$$

The witness N_ρ is a whole finite index. Nothing in the definition requires a final stage after which no further stages exist.

Lemma 55.9 (Resolution monotonicity)**.** *If X is eventually contained at resolution ρ around L and $\rho \leq \sigma$, then X is eventually contained at resolution σ around L.*

Proof. The same witness index works because

$$e_n(X; L) < \rho \leq \sigma.$$

□

The reverse implication is not generally valid. Satisfying a coarse resolution does not certify a finer one.

55.4 Constructive convergence

Definition 55.10 (Constructive convergence to a retained target)**.** *An approximation family $X = (X_n)$* converges constructively *to $L \in \mathbb{Q}_{\mathrm{C}}$ when, for every positive resolution report ρ, there exists a finite witness index N_ρ such that*

$$n \geq N_\rho \quad \Longrightarrow \quad d_{\mathrm{C}}(X_n, L) < \rho.$$

Write

$$X_n \longrightarrow_{\mathrm{C}} L$$

or, as a later compressed report,

$$\lim_{n \to \infty} X_n = L.$$

The infinity mark in the compressed notation does not denote a final natural index. It reports that the tail condition is asserted for every finite stage beyond a suitable finite witness.

Definition 55.11 (Convergence certificate)**.** *A* convergence certificate *for $X_n \to_{\mathrm{C}} L$ consists of:*

(i) the finite rule generating X_n;

(ii) the retained target report L;

(iii) the positive-resolution grammar;

(iv) a witness-selection rule $\rho \mapsto N_\rho$ or a proof that such a finite witness exists;

(v) and a finite verification that every admitted tail stage satisfies the requested bound.

A bare limit symbol suppresses this ledger. Primitive Constructive Logic treats the suppressed data as recoverable obligations, not as nonexistent history.

Principle 55.12 (No retrojected completion)**.** *The statement $X_n \to_{\mathrm{C}} L$ is a rule-level relation among finite reports. It does not imply that the approximation family has already been replaced by a completed infinite object, and it does not make any finite stage constructively identical to L.*

55.5 Exact equality, stabilization, and convergence

Three different closure conditions must be separated.

Definition 55.13 (Finite-stage equality)**.** *The family has* finite-stage equality *with L at index n when*

$$X_n =_{\mathrm{C}} L.$$

Definition 55.14 (Eventual stabilization)**.** *The family* eventually stabilizes *at L when there exists a finite N such that*

$$n \geq N \quad \Longrightarrow \quad X_n =_{\mathrm{C}} L.$$

Theorem 55.15 (Stabilization implies convergence)**.** *If X eventually stabilizes at L, then $X_n \to_{\mathrm{C}} L$.*

Proof. Beyond the stabilization index, every residual is the typed exact-zero report, whose magnitude is smaller than every positive resolution. □

The converse fails.

Example 55.16 (Positive finite residuals converging to zero)**.** *For each positive natural index n, let*

$$X_n = \frac{1}{n}.$$

Every finite stage is a positive ratio report, so

$$X_n \neq_{\mathrm{C}} 0$$

for every finite n.

Let the requested positive resolution be

$$\rho = \frac{a}{b}, \qquad a, b \in \mathbb{N}, \quad a > 0, \quad b > 0.$$

Choose

$$N_\rho = b + 1.$$

For every $n \geq b + 1$,

$$0 < \frac{1}{n} \leq \frac{1}{b+1} < \frac{a}{b},$$

because $a \geq 1$. Hence

$$\frac{1}{n} \longrightarrow_{\mathrm{C}} 0.$$

No numerator occurrence became literal absence. The family satisfies every requested positive resolution while retaining a positive numerator at each finite stage.

Theorem 55.17 (Fixed-positive nonvanishing)**.** *Let $X = (X_n)$ and L be given. Suppose there exists a fixed positive ratio δ such that, beyond every proposed finite index N, some later stage satisfies*

$$d_{\mathrm{C}}(X_n, L) \geq \delta.$$

Then X_n does not converge constructively to L.

Proof. Request the positive resolution $\rho = \delta$. No tail can remain strictly inside that resolution, because every tail contains a stage whose residual magnitude is at least δ. □

Corollary 55.18 (A fixed positive report is not zero by refinement)**.** *If $c > 0$ is fixed and $X_n = c$ for every n, then X_n does not converge to zero.*

This blocks the informal substitution

$$\text{“arbitrarily refined”} \quad \rightsquigarrow \quad \text{“a fixed positive primitive became zero.”}$$

55.6 Orientation of approach

Convergence controls residual magnitude. It does not, by itself, erase residual orientation.

Definition 55.19 (Eventual approach from below)**.** *A family approaches L* eventually from below *when it converges to L and there exists N such that*

$$n \geq N \quad \Longrightarrow \quad X_n \preceq_{\mathbb{Q}_{\mathrm{C}}} L.$$

Definition 55.20 (Eventual approach from above)**.** *A family approaches L* eventually from above *when it converges to L and there exists N such that*

$$n \geq N \quad \Longrightarrow \quad L \preceq_{\mathbb{Q}_{\mathrm{C}}} X_n.$$

A convergent family need satisfy neither one-sided condition.

Example 55.21 (Alternating orientation with contracting magnitude)**.** *Let*

$$X_n = \frac{(-1)^n}{n}$$

for positive n. The residual orientation alternates at every stage, but

$$d_{\mathrm{C}}(X_n, 0) = \frac{1}{n} \longrightarrow_{\mathrm{C}} 0.$$

Therefore $X_n \to_{\mathrm{C}} 0$ while the approach route does not stabilize to one orientation.

The scalar limit report forgets this alternating orientation unless the approach ledger is retained separately.

55.7 Uniqueness of a retained limit

Theorem 55.22 (Uniqueness of constructive limit in $\mathbb{Q}_{\mathrm{C}}$)**.** *If*

$$X_n \longrightarrow_{\mathrm{C}} L \qquad \textit{and} \qquad X_n \longrightarrow_{\mathrm{C}} M,$$

then

$$L =_{\mathbb{Q}_{\mathrm{C}}} M.$$

Proof. Assume instead that

$$d := d_{\mathrm{C}}(L, M) > 0.$$

Choose the positive resolution

$$\rho = \frac{d}{3}.$$

There are finite witnesses N_L and N_M such that beyond their maximum,

$$d_{\mathrm{C}}(X_n, L) < \frac{d}{3} \qquad \text{and} \qquad d_{\mathrm{C}}(X_n, M) < \frac{d}{3}.$$

The finite residual triangle law gives

$$d = d_{\mathrm{C}}(L, M) \le d_{\mathrm{C}}(L, X_n) + d_{\mathrm{C}}(X_n, M) < \frac{2d}{3},$$

a contradiction. Therefore $d = 0$, so the two targets are equal as normalized signed-ratio reports. □

Limit uniqueness is thus a theorem of the separated finite residual grammar. It is not supplied by the notation lim itself.

55.8 Algebraic preservation of limits

Limit arithmetic must be proved from finite residual bounds. It cannot be licensed merely by symbolic substitution.

Theorem 55.23 (Sum, inverse, and difference laws)**.** *Suppose*

$$X_n \to_{\mathrm{C}} L \qquad \textit{and} \qquad Y_n \to_{\mathrm{C}} M.$$

Then

$$X_n + Y_n \to_{\mathrm{C}} L + M,$$
$$-X_n \to_{\mathrm{C}} -L,$$

and

$$X_n - Y_n \to_{\mathrm{C}} L - M.$$

Proof. Given positive resolution ρ, choose tails satisfying

$$d_{\mathrm{C}}(X_n, L) < \frac{\rho}{2}, \qquad d_{\mathrm{C}}(Y_n, M) < \frac{\rho}{2}.$$

The residual triangle law gives

$$d_{\mathrm{C}}(X_n + Y_n, L + M) \le d_{\mathrm{C}}(X_n, L) + d_{\mathrm{C}}(Y_n, M) < \rho.$$

Orientation reversal preserves magnitude, proving the inverse law. The difference law follows by combining the first two. □

Theorem 55.24 (Fixed scaling law)**.** *Let $C \in \mathbb{Q}_{\mathrm{C}}$ be fixed. If $X_n \to_{\mathrm{C}} L$, then*

$$CX_n \to_{\mathrm{C}} CL.$$

Proof. If $C = 0$, every product is an exact typed zero. If $C \neq 0$, request resolution $\rho/\|C\|_{\mathrm{C}}$ for the residual of X_n from L. Finite ratio multiplication then gives

$$d_{\mathrm{C}}(CX_n, CL) = \|C\|_{\mathrm{C}}\, d_{\mathrm{C}}(X_n, L) < \rho.$$

□

Lemma 55.25 (Convergent families are eventually bounded)**.** *If $X_n \to_{\mathrm{C}} L$, then there exist a positive finite bound B and a finite index N such that*

$$n \geq N \quad \Longrightarrow \quad \|X_n\|_{\mathrm{C}} \leq B.$$

Proof. Use resolution 1. Beyond its witness index,

$$d_{\mathrm{C}}(X_n, L) < 1.$$

The residual triangle law yields

$$\|X_n\|_{\mathrm{C}} \leq \|L\|_{\mathrm{C}} + 1.$$

Take $B = \|L\|_{\mathrm{C}} + 1$. □

Theorem 55.26 (Product law)**.** *If $X_n \to_{\mathrm{C}} L$ and $Y_n \to_{\mathrm{C}} M$, then*

$$X_nY_n \to_{\mathrm{C}} LM.$$

Proof. Choose an eventual bound B for $\|X_n\|_{\mathrm{C}}$ and set

$$K = B + \|M\|_{\mathrm{C}} + 1.$$

Given positive resolution ρ, choose a common tail on which

$$d_{\mathrm{C}}(Y_n, M) < \frac{\rho}{K} \qquad \text{and} \qquad d_{\mathrm{C}}(X_n, L) < \frac{\rho}{K}.$$

Using the finite decomposition

$$X_nY_n - LM = X_n(Y_n - M) + M(X_n - L),$$

finite magnitude and triangle laws give

$$\begin{aligned} d_{\mathrm{C}}(X_nY_n, LM) &\leq \|X_n\|_{\mathrm{C}} d_{\mathrm{C}}(Y_n, M) + \|M\|_{\mathrm{C}} d_{\mathrm{C}}(X_n, L) \\ &< \frac{B\rho}{K} + \frac{\|M\|_{\mathrm{C}}\rho}{K} < \rho. \end{aligned}$$

□

Theorem 55.27 (Reciprocal law away from zero)**.** *If $X_n \to_{\mathrm{C}} L$ and $L \neq 0$, then there exists a finite index after which every X_n is nonzero and*

$$\frac{1}{X_n} \longrightarrow_{\mathrm{C}} \frac{1}{L}.$$

Proof. Choose resolution $\|L\|_{\mathrm{C}}/2$. Beyond its witness index,

$$d_{\mathrm{C}}(X_n, L) < \frac{\|L\|_{\mathrm{C}}}{2},$$

so

$$\|X_n\|_{\mathrm{C}} > \frac{\|L\|_{\mathrm{C}}}{2} > 0.$$

Thus the reciprocal compiler is defined on the tail. The finite identity

$$\frac{1}{X_n} - \frac{1}{L} = \frac{L - X_n}{X_n L}$$

gives

$$d_{\mathrm{C}}\Big(\frac{1}{X_n}, \frac{1}{L}\Big) \le \frac{2\, d_{\mathrm{C}}(X_n, L)}{\|L\|_{\mathrm{C}}^2}.$$

For any requested output resolution ρ, require

$$d_{\mathrm{C}}(X_n, L) < \frac{\rho \|L\|_{\mathrm{C}}^2}{2}$$

on a sufficiently late common tail. The displayed bound then yields the desired reciprocal resolution. □

Every law above is a fiber-preservation theorem for the limit report. None licenses replacement of a finite approximation ledger by its target when stagewise source information matters.

55.9 Nested whole bounds

Convergence may be certified by whole finite upper and lower reports rather than by a single displayed error formula.

Definition 55.28 (Nested bound family)**.** *A* nested bound family *is a pair of finite-ratio families*

$$(A_m, B_m)_{m \in \mathbb{N}}$$

satisfying

$$A_m \preceq A_{m+1} \preceq B_{m+1} \preceq B_m.$$

Its width report is

$$W_m := B_m - A_m \ge 0.$$

Every stage is a whole finite interval report with two finite endpoints. No endpoint is a partial object, and no interval width is presumed to vanish at a finite stage.

Definition 55.29 (Target retained by nested bounds)**.** *A target $L \in \mathbb{Q}_{\mathrm{C}}$ is retained by the nested bound family when*

$$A_m \preceq L \preceq B_m$$

for every finite m.

Theorem 55.30 (Nested-bound convergence)**.** *Suppose L is retained by (A_m, B_m) and*

$$W_m \longrightarrow_{\mathrm{C}} 0.$$

If a family X_m satisfies

$$A_m \preceq X_m \preceq B_m$$

for every m, then

$$X_m \longrightarrow_{\mathrm{C}} L.$$

Proof. Both X_m and L lie inside the same finite bound carrier, so

$$d_{\mathrm{C}}(X_m, L) \leq W_m.$$

Every requested positive resolution eventually dominates W_m, and hence also the residual magnitude. □

Principle 55.31 (Bounds without a retained target)**.** *A nested rational bound family whose widths converge to zero need not yet possess a target inside $\mathbb{Q}_{\mathrm{C}}$. Without such a retained target, it is a completion specification, not a proved rational limit.*

This distinction prevents the completeness of the real carrier from being smuggled into the definition of rational convergence.

55.10 The constructive squeeze theorem

Theorem 55.32 (Squeeze by retained bounds)**.** *Suppose*

$$A_n \preceq X_n \preceq B_n$$

for every sufficiently late stage and

$$A_n \to_{\mathrm{C}} L, \qquad B_n \to_{\mathrm{C}} L.$$

Then

$$X_n \to_{\mathrm{C}} L.$$

Proof. For a requested positive resolution ρ, choose a tail on which both bounding residuals from L are smaller than ρ. Since X_n lies between the two bounds, its signed residual from L cannot exceed the larger bounding residual magnitude. Hence it is also smaller than ρ on that tail. □

The theorem is a finite order argument repeated at every requested resolution. It does not require a geometric picture of continuously shrinking intervals.

55.11 Cauchy tail coherence

A family may exhibit internally contracting tails even when no target has yet been constructed in the present carrier.

Definition 55.33 (Constructive Cauchy family)**.** *An approximation family $X = (X_n)$ in $\mathbb{Q}_{\mathrm{C}}$ is* Cauchy *when, for every positive resolution ρ, there exists a finite index N_ρ such that*

$$m, n \geq N_\rho \quad \Longrightarrow \quad d_{\mathrm{C}}(X_m, X_n) < \rho.$$

The Cauchy condition compares finite tail stages with one another. It does not assert that a target already exists in the carrier.

Theorem 55.34 (Convergence implies Cauchy coherence)**.** *Every constructively convergent family in $\mathbb{Q}_{\mathrm{C}}$ is Cauchy.*

Proof. Suppose $X_n \to_{\mathrm{C}} L$. For requested resolution ρ, choose a tail on which

$$d_{\mathrm{C}}(X_n, L) < \frac{\rho}{2}.$$

For m, n in that tail, the residual triangle law gives

$$d_{\mathrm{C}}(X_m, X_n) \le d_{\mathrm{C}}(X_m, L) + d_{\mathrm{C}}(L, X_n) < \rho.$$

□

Principle 55.35 (No converse before completion)**.** *The assertion that every Cauchy family has a target is not a theorem of the finite rational carrier constructed so far. It is the defining obligation of a later completion.*

Definition 55.36 (Cauchy zero-distance equivalence)**.** *For Cauchy families $X = (X_n)$ and $Y = (Y_n)$, write*

$$X \sim_{\mathrm{Cau}} Y$$

when

$$d_{\mathrm{C}}(X_n, Y_n) \longrightarrow_{\mathrm{C}} 0.$$

Equivalently, for every positive resolution ρ, there is a finite N_ρ such that

$$n \ge N_\rho \quad \Longrightarrow \quad d_{\mathrm{C}}(X_n, Y_n) < \rho.$$

Theorem 55.37 (Cauchy zero-distance is an equivalence relation)**.** *The relation $\sim_{\mathrm{Cau}}$ is reflexive, symmetric, and transitive on the family of Cauchy approximation rules.*

Proof. Reflexivity follows from exact zero residual at every synchronized stage. Symmetry follows from residual symmetry. For transitivity, if $X \sim_{\mathrm{Cau}} Y$ and $Y \sim_{\mathrm{Cau}} Z$, request half of any desired resolution for each relation and apply the residual triangle law. □

This equivalence relation is the natural candidate for the completion quotient in the next section. At this stage it is only an explicitly constructed relation among finite approximation families.

55.12 Limit fibers and preservation obligations

Let $\mathcal{D}_{\mathbb{Q}}$ denote the family of approximation rules that converge to retained targets in $\mathbb{Q}_{\mathrm{C}}$. Uniqueness permits the limit-value map

$$\mathrm{Lim} : \mathcal{D}_{\mathbb{Q}} \longrightarrow \mathbb{Q}_{\mathrm{C}}, \qquad X \longmapsto L.$$

The fiber

$$\mathrm{Lim}^{-1}(L)$$

contains every admitted approximation family whose limit report is L. These families may differ in source, formula, recurrence, rate, side of approach, oscillation, finite-stage zeros, and proof certificate.

Example 55.38 (One limit fiber, different stagewise constructions)**.** *The constant-zero family*

$$Z_n = 0$$

and the positive family

$$X_n = \frac{1}{n}$$

both lie in the zero-limit fiber:

$$\mathrm{Lim}(Z) = 0 = \mathrm{Lim}(X).$$

Yet:

(i) every stage of Z is an exact typed zero;

(ii) no stage of X is zero;

(iii) Z stabilizes immediately;

(iv) X never stabilizes;

(v) their source and residual ledgers are different.

Theorem 55.39 (Limit-fiber preservation condition)**.** *A property P of approximation families can be computed from the limit value alone exactly when P is constant on every fiber of* Lim*.*

Proof. This is the projection-fiber factorization theorem of Section 54 applied to the limit-value map. □

Therefore properties such as "has limit zero" descend to the limit report, while properties such as "contains an exact zero stage," "approaches from above," "stabilizes," or "uses a particular recurrence" generally do not.

Principle 55.40 (Limit equality is typed equality)**.** *From*

$$\lim X_n = \lim Y_n$$

one may infer equality of limit values. One may not infer constructive equality of the approximation families unless an additional theorem proves that the relevant source property is constant on the common limit fiber.

55.13 Finite prefixes do not determine convergence

Theorem 55.41 (Finite-prefix underdetermination)**.** *No finite prefix by itself determines a unique limit or even determines whether an unrestricted continuation converges.*

Proof. Let

$$[X_0, X_1, \ldots, X_N]$$

be any completed finite prefix. It admits, among others:

(i) a continuation that is constantly 0 after N, which converges to 0;

(ii) a continuation that is constantly 1 after N, which converges to 1;

(iii) a continuation alternating between 0 and 1, which does not converge.

All three preserve the given finite prefix. Therefore convergence belongs to the retained continuation rule and its proof, not to the prefix alone. □

A finite descriptor may nevertheless certify convergence when its rule implies the required tail bounds. The theorem blocks only unsupported extrapolation from a finite sample.

55.14 Radix limits and the report $0.999\ldots$

For each positive natural m, define the finite decimal prefix

$$p_m := \sum_{k=1}^{m} 9 \cdot 10^{-k}.$$

Finite induction using the already constructed power and ratio laws gives

$$p_m = 1 - 10^{-m}.$$

Hence the exact finite residual is

$$1 - p_m = 10^{-m} > 0.$$

Therefore

$$p_m \neq_{\mathrm{C}} 1$$

for every finite m.

Theorem 55.42 (Decimal-nine prefix convergence)**.** *The finite prefix family satisfies*

$$p_m \longrightarrow_{\mathrm{C}} 1.$$

Proof. Let the requested resolution be $\rho = a/b > 0$. Choose $m = b$. Since $10^b > b$ for every positive natural b,

$$10^{-b} < \frac{1}{b} \leq \frac{a}{b}.$$

For every $n \geq b$,

$$d_{\mathrm{C}}(p_n, 1) = 10^{-n} \leq 10^{-b} < \rho.$$

□

There are now three distinct report types:

(i) a finite prefix p_m, which is never equal to 1;

(ii) the finite recurrence descriptor "append another 9 digit," which is a rule object;

(iii) the limit-value report of the generated prefix family, which equals 1.

Principle 55.43 (Typed interpretation of recurring notation)**.** *If the notation*

$$0.999\ldots_{10}$$

is declared to denote the finite recurrence descriptor, then it is not constructively identical to the unit report. If it is declared to denote the limit value of its finite-prefix family, then

$$0.999\ldots_{10} =_{\lim} 1.$$

The equality is a limit-value equality, not equality of any finite prefix with 1 *and not equality of the source descriptor with the unit construction.*

This resolves the apparent conflict without erasing either finite residuals or the proved limit value.

55.15 Divergence and nonconvergence witnesses

Failure to observe stabilization is not itself a proof of divergence. Nonconvergence requires a witness showing that at least one required resolution condition fails for every proposed target.

Definition 55.44 (Resolution-separated tail behavior)**.** *A family has* resolution-separated tail behavior *when there exists a positive δ such that, beyond every finite index, two later stages m, n satisfy*

$$d_{\mathrm{C}}(X_m, X_n) \geq \delta.$$

Theorem 55.45 (Separated tails cannot converge)**.** *A family with resolution-separated tail behavior is not constructively convergent.*

Proof. Every convergent family is Cauchy. Request resolution δ in the Cauchy condition. The separated-tail witness contradicts the existence of any tail whose pairwise residuals are all strictly smaller than δ. □

Example 55.46 (Persistent alternation)**.** *The family*

$$X_n = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd} \end{cases}$$

has later stages separated by residual magnitude 1 *beyond every finite index. It therefore does not converge.*

The conclusion follows from a retained positive separation, not from a picture of oscillation on a pre-existing coordinate line.

55.16 Cofinal resolution families and base independence

Testing every positive ratio is conceptually exact but can be compressed when a declared resolution family refines beneath every positive ratio.

Definition 55.47 (Cofinal resolution family toward zero)**.** *A family $\mathcal{E}$ of positive ratio reports is* cofinal toward zero *when, for every positive ratio ρ, some $\varepsilon \in \mathcal{E}$ satisfies*

$$0 < \varepsilon < \rho.$$

Theorem 55.48 (Cofinal resolution test)**.** *If $\mathcal{E}$ is cofinal toward zero, then $X_n \to_{\mathrm{C}} L$ exactly when eventual resolution containment holds for every $\varepsilon \in \mathcal{E}$.*

Proof. The forward direction is immediate because every ε is a positive resolution. Conversely, given arbitrary positive ρ, choose $\varepsilon \in \mathcal{E}$ with $\varepsilon < \rho$. A tail lying within ε also lies within ρ by resolution monotonicity. □

Theorem 55.49 (Radix-power resolutions are cofinal)**.** *For every finite base $b \geq 2$, the family*

$$\mathcal{E}_b = \{b^{-m} : m \in \mathbb{N},\ m > 0\}$$

is cofinal toward zero in the positive constructive ratios.

Proof. Let $\rho = a/c > 0$. Choose $m = c$. Since $b \geq 2$ and $c > 0$,

$$b^c > c.$$

Therefore

$$b^{-c} < \frac{1}{c} \leq \frac{a}{c} = \rho.$$

□

Thus binary, decimal, or any other finite-base resolution ladder tests the same convergence relation when the base and comparison theorem are retained. No radix is ontologically privileged by the definition of convergence.

55.17 No positive finite infinitesimal in the rational carrier

Theorem 55.50 (No fixed positive rational lies below every positive resolution)**.** *For every fixed $\delta \in \mathbb{Q}_C$ with $\delta > 0$, there exists a positive resolution ρ such that*

$$0 < \rho < \delta.$$

Proof. Take $\rho = \delta/2$. □

Corollary 55.51 (No finite positive rational is an undetectable residual)**.** *A fixed positive rational residual cannot satisfy all positive-resolution tests against zero.*

Convergence to zero therefore requires a family of changing finite residuals whose refinement responds to every requested positive bound. It cannot be represented by one fixed positive rational relabeled as zero.

55.18 Limit preservation theorem

Theorem 55.52 (Constructive limit preservation)**.** *A limit-value report preserves exactly the properties of an approximation family that are constant on the corresponding limit fiber. In particular, the limit value alone generally does not preserve:*

(i) the finite rule or recurrence generating the family;

(ii) the source object at each index;

(iii) the sign or side of approach;

(iv) the rate at which requested resolutions are met;

(v) the witness-selection rule $\rho \mapsto N_\rho$;

(vi) whether any finite stage equals the target;

(vii) whether the family eventually stabilizes;

(viii) the base or positional grammar of the approximations;

(ix) or the proof route establishing convergence.

These data may be discarded only for a later property proved constant on the limit fiber.

The limit map is therefore another quotient report. It is useful precisely because it suppresses distinctions irrelevant to limit-value arithmetic, but it cannot be used to prove that those distinctions never existed.

55.19 Dependency summary and handoff to completion

The construction of convergence is now

$$\begin{aligned}\text{finite signed ratios} &\longrightarrow \text{finite residual and magnitude} \longrightarrow \text{positive resolution report}\\ &\longrightarrow \text{eventual tail containment} \longrightarrow \text{constructive convergence}\\ &\longrightarrow \text{limit-value fibers and preservation audit.}\end{aligned}$$

The essential distinctions are:

(1) finite-stage equality is not convergence;

(2) eventual stabilization implies convergence but is not required by it;

(3) a positive residual may survive at every finite stage while its family converges to zero;

(4) no fixed positive finite ratio becomes zero through refinement;

(5) convergence to a retained target implies Cauchy tail coherence;

(6) Cauchy coherence alone does not yet construct a target in $\mathbb{Q}_{\mathrm{C}}$;

(7) equal limit values do not imply equal approximation families;

(8) a finite prefix does not determine convergence without a continuation rule;

(9) recurring notation must distinguish its finite descriptor from its limit-value projection;

(10) and every operation on future completed values must be proved invariant under Cauchy zero-distance equivalence.

The next section should construct the completed number carrier from Cauchy approximation families modulo $\sim_{\mathrm{Cau}}$. It must prove that addition, multiplication, signed order, zero, and equality are well defined on the resulting fibers, and it must state exactly which finite approximation data the completion quotient forgets.

56 Rational completion and real-number reports

The finite rational carrier already constructed in this manuscript contains signed ratio reports and their finite arithmetic. It does not contain a target for every Cauchy approximation family. Completion is the declared operation that supplies those missing targets while preserving exactly the relations proved constant on Cauchy zero-distance fibers.

56.1 The finite rational carrier and the completion obligation

Let

$$\mathbb{Q}_{\mathrm{C}}$$

denote the carrier of finite signed constructive ratios, modulo the ratio equality established earlier. Every member of $\mathbb{Q}_{\mathrm{C}}$ has at least one finite numerator-denominator construction, although distinct constructions may share the same rational report.

Let

$$\mathrm{Cau}(\mathbb{Q}_{\mathrm{C}})$$

denote the report family of all declared Cauchy approximation families

$$X = (X_n)_{n\in\mathbb{N}}$$

with $X_n \in \mathbb{Q}_{\mathrm{C}}$ at every finite stage.

Principle 56.1 (Completion obligation)**.** *The finite rational carrier is not declared complete. A Cauchy family may fail to converge to any retained rational target. A completed carrier must therefore do more than rename rational limits: it must admit a report for every Cauchy family and identify two such reports exactly when their stagewise residual tends to zero.*

The necessary equivalence relation has already been constructed:

$$X \sim_{\mathrm{Cau}} Y \quad\Longleftrightarrow\quad d_{\mathrm{C}}(X_n, Y_n) \longrightarrow_{\mathrm{C}} 0.$$

56.2 Completed real reports

Definition 56.2 (Completed real carrier)**.** *The* completed real carrier *is the quotient report*

$$\mathbb{R}_{\mathrm{C}} := \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}})/\sim_{\mathrm{Cau}}.$$

A completed real report *is an equivalence class*

$$[X] = \{Y \in \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}}) : Y \sim_{\mathrm{Cau}} X\}.$$

The brackets do not assert that the source family has vanished. They mark the output of a quotient map

$$\kappa : \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}}) \longrightarrow \mathbb{R}_{\mathrm{C}}, \qquad X \longmapsto [X].$$

Definition 56.3 (Completed-real equality)**.** *For Cauchy families X and Y, define*

$$[X] =_{\mathbb{R}_{\mathrm{C}}} [Y] \quad\Longleftrightarrow\quad X \sim_{\mathrm{Cau}} Y.$$

Thus completed-real equality is neither finite-stage identity nor equality of generating rules. It is equality under the declared Cauchy completion quotient.

Theorem 56.4 (Every Cauchy family has a completed target)**.** *For every $X \in \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}})$, the class $[X]$ is a member of $\mathbb{R}_{\mathrm{C}}$ and X converges to $[X]$ in the completed carrier.*

Proof. The first statement is the quotient construction. For the second, compare the nth rational stage X_n, embedded as a constant family, with the class $[X]$. Cauchy tail coherence says that for every positive rational resolution ρ, there is N_ρ such that

$$m, n \geq N_\rho \Longrightarrow d_{\mathrm{C}}(X_m, X_n) < \rho.$$

Fixing $n \geq N_\rho$ and allowing m to range over the tail shows that the constant report X_n lies within ρ of the completed class represented by X. Hence the embedded stages converge to $[X]$. □

The theorem is not a discovery of a previously assumed point on a continuum. It states what target the completion quotient assigns to a coherent finite approximation family.

56.3 Embedding finite rationals

Definition 56.5 (Constant-family embedding)**.** *For $q \in \mathbb{Q}_{\mathrm{C}}$, let*

$$\underline{q} = (q, q, q, \ldots)$$

denote the constant Cauchy family. Define

$$\iota : \mathbb{Q}_{\mathrm{C}} \longrightarrow \mathbb{R}_{\mathrm{C}}, \qquad \iota(q) := [\underline{q}].$$

Theorem 56.6 (The rational embedding is injective)**.** *If*

$$\iota(p) =_{\mathbb{R}_{\mathrm{C}}} \iota(q),$$

then

$$p = q$$

in $\mathbb{Q}_{\mathrm{C}}$.

Proof. Equality of the completed reports means

$$d_{\mathrm{C}}(p, q) \longrightarrow_{\mathrm{C}} 0$$

for the constant residual family. If $p \neq q$, then the residual magnitude is one fixed positive rational. The fixed-positive nonvanishing theorem from Section 55 says that such a family cannot converge to zero. Hence $p = q$. □

The embedding therefore preserves finite rational distinction. From this point onward, a rational may be written inside $\mathbb{R}_{\mathrm{C}}$ without displaying ι when the carrier is clear, but the embedding remains part of the dependency ledger.

Theorem 56.7 (Rational limits agree with the embedding)**.** *If a rational Cauchy family X converges in $\mathbb{Q}_{\mathrm{C}}$ to a retained rational q, then*

$$[X] =_{\mathbb{R}_{\mathrm{C}}} \iota(q).$$

Proof. Rational convergence to q is exactly the statement

$$d_{\mathrm{C}}(X_n, q) \longrightarrow_{\mathrm{C}} 0,$$

which says $X \sim_{\mathrm{Cau}} \underline{q}$. □

56.4 Nested rational intervals as completion reports

Cauchy families are not the only finite description of a completed real. Nested rational bounds provide an equivalent route when their widths converge to zero.

Definition 56.8 (Cauchy interval specification)**.** *A* Cauchy interval specification *is a family of rational endpoint pairs*

$$I_n = [L_n, U_n]$$

satisfying

$$L_n \leq L_{n+1} \leq U_{n+1} \leq U_n$$

and

$$U_n - L_n \longrightarrow_{\mathrm{C}} 0.$$

Every stage is a finite rational interval report. The interval need not collapse at any finite stage.

Theorem 56.9 (Nested endpoints are equivalent Cauchy families)**.** *For a Cauchy interval specification (I_n), both endpoint families (L_n) and (U_n) are Cauchy and*

$$(L_n) \sim_{\mathrm{Cau}} (U_n).$$

Proof. For $m \geq n$, nesting gives

$$L_n \leq L_m \leq U_m \leq U_n.$$

Therefore

$$0 \leq L_m - L_n \leq U_n - L_n$$

and similarly

$$0 \leq U_n - U_m \leq U_n - L_n.$$

Since the width tends to zero, each endpoint family is Cauchy. Also

$$0 \leq U_n - L_n \longrightarrow_{\mathrm{C}} 0,$$

so the two endpoint families are Cauchy-equivalent. □

Definition 56.10 (Real report retained by nested intervals)**.** *The completed real retained by (I_n) is*

$$\mathrm{Real}(I) := [(L_n)] = [(U_n)].$$

Theorem 56.11 (Interior choice invariance)**.** *If $X_n \in \mathbb{Q}_{\mathrm{C}}$ satisfies*

$$L_n \leq X_n \leq U_n$$

for every n, then (X_n) is Cauchy and

$$[(X_n)] = \mathrm{Real}(I).$$

Proof. Every interior choice lies within the interval width of either endpoint:

$$0 \leq X_n - L_n \leq U_n - L_n.$$

The width tends to zero, so $X \sim_{\mathrm{Cau}} L$. Cauchy coherence follows from equivalence with the Cauchy endpoint family. □

The completed value therefore depends on the shrinking interval fiber, not on which finite point is selected from each interval.

56.5 Every Cauchy family admits nested rational bounds

Theorem 56.12 (Nested-bound representative theorem)**.** *Every rational Cauchy family $X = (X_n)$ admits a Cauchy interval specification whose retained completed real is $[X]$.*

Proof. Use the cofinal dyadic resolution ladder. Choose strictly increasing indices N_k such that

$$m, n \geq N_k \Longrightarrow d_{\mathrm{C}}(X_m, X_n) < 2^{-(k+2)}.$$

Set

$$c_k := X_{N_k}, \qquad r_k := 2^{-(k+1)}, \qquad I_k := [c_k - r_k, c_k + r_k].$$

Because $N_{k+1} \geq N_k$,

$$|c_{k+1} - c_k| < 2^{-(k+2)} = r_{k+1}.$$

Hence every point of I_{k+1} lies within

$$r_{k+1} + |c_{k+1} - c_k| < 2r_{k+1} = r_k$$

of c_k, so

$$I_{k+1} \subseteq I_k.$$

The widths are

$$2r_k = 2^{-k} \longrightarrow_{\mathrm{C}} 0.$$

Finally, the center subsequence (c_k) is equivalent to X, because a Cauchy family and any cofinal tail subsequence have zero Cauchy distance. Therefore the nested intervals retain $[X]$. □

This theorem shows that Cauchy-family completion and nested-rational-interval completion report the same carrier once their equivalence has been proved.

56.6 Addition and additive orientation on completed reports

For Cauchy families $X = (X_n)$ and $Y = (Y_n)$, define stagewise families

$$X + Y := (X_n + Y_n), \qquad -X := (-X_n).$$

Lemma 56.13 (Stagewise sums and negatives are Cauchy)**.** *If X and Y are rational Cauchy families, then $X + Y$ and $-X$ are rational Cauchy families.*

Proof. For requested resolution ρ, place each input tail within $\rho/2$. Then

$$|(X_m + Y_m) - (X_n + Y_n)| \leq |X_m - X_n| + |Y_m - Y_n| < \rho.$$

Negation preserves residual magnitude exactly. □

Theorem 56.14 (Addition and negation descend to completion)**.** *The definitions*

$$[X] + [Y] := [X + Y], \qquad -[X] := [-X]$$

are independent of the chosen representatives.

Proof. Suppose $X \sim_{\mathrm{Cau}} X'$ and $Y \sim_{\mathrm{Cau}} Y'$. Then

$$|(X_n + Y_n) - (X'_n + Y'_n)| \leq |X_n - X'_n| + |Y_n - Y'_n| \longrightarrow_{\mathrm{C}} 0.$$

Thus $X + Y \sim_{\mathrm{Cau}} X' + Y'$. Negation preserves the same residual magnitude, so $-X \sim_{\mathrm{Cau}} -X'$. □

The additive zero and subtraction are then

$$0_{\mathbb{R}_{\mathrm{C}}} := [\underline{0}], \qquad [X] - [Y] := [X] + (-[Y]).$$

56.7 Boundedness of rational Cauchy families

Multiplication requires a bound because a small factor error can be amplified by an unbounded companion.

Lemma 56.15 (Cauchy boundedness)**.** *Every rational Cauchy family $X = (X_n)$ has a finite rational bound $M_X > 0$ such that*

$$|X_n| \le M_X$$

for every finite n.

Proof. Choose a tail index N such that

$$m, n \ge N \Longrightarrow |X_m - X_n| < 1.$$

Then every tail term satisfies

$$|X_m| \le |X_N| + 1.$$

The finite initial list

$$|X_0|, \ldots, |X_{N-1}|$$

has a largest retained rational magnitude. Choose M_X larger than that finite maximum and larger than $|X_N| + 1$. □

The bound is not unique and is not part of the completed scalar unless explicitly retained. Its role is to certify preservation of multiplication under the completion quotient.

56.8 Multiplication on completed reports

For Cauchy families X and Y, define the stagewise product

$$XY := (X_n Y_n).$$

Lemma 56.16 (Stagewise products are Cauchy)**.** *If X and Y are rational Cauchy families, then XY is rational Cauchy.*

Proof. Choose finite bounds $M_X, M_Y \ge 1$. For m, n,

$$|X_m Y_m - X_n Y_n| \le |X_m|\,|Y_m - Y_n| + |Y_n|\,|X_m - X_n|.$$

Given positive resolution ρ, choose a common tail on which

$$|Y_m - Y_n| < \frac{\rho}{2M_X} \quad \text{and} \quad |X_m - X_n| < \frac{\rho}{2M_Y}.$$

The displayed residual is then smaller than ρ. □

Theorem 56.17 (Multiplication descends to completion)**.** *The definition*

$$[X] \cdot [Y] := [XY]$$

is independent of the chosen representatives.

Proof. Suppose $X \sim_{\mathrm{Cau}} X'$ and $Y \sim_{\mathrm{Cau}} Y'$. Choose rational bounds M_X for X and $M_{Y'}$ for Y'. Then

$$|X_nY_n - X'_nY'_n| \le |X_n|\,|Y_n - Y'_n| + |Y'_n|\,|X_n - X'_n|.$$

Both right-hand residual families tend to zero, and the finite bounds prevent amplification from defeating any requested resolution. Hence

$$XY \sim_{\mathrm{Cau}} X'Y'.$$

□

Corollary 56.18 (The rational embedding preserves arithmetic)**.** *For $p, q \in \mathbb{Q}_{\mathrm{C}}$,*

$$\iota(p+q) = \iota(p) + \iota(q), \qquad \iota(pq) = \iota(p)\iota(q), \qquad \iota(-p) = -\iota(p).$$

At this point $\mathbb{R}_{\mathrm{C}}$ has completed addition, subtraction, and multiplication. Reciprocal construction for completed nonzero reports will require the signed separation and order developed in Section 58.

56.9 Completed zero versus finite exact zero

Definition 56.19 (Completed zero class)**.** *The completed zero is*

$$0_{\mathbb{R}_{\mathrm{C}}} := [\underline{0}] = \{X \in \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}}) : X_n \to_{\mathrm{C}} 0\}.$$

A member of the zero class need not contain a finite exact-zero stage.

Example 56.20 (Positive stages in the completed-zero fiber)**.** *The family*

$$X_n = \frac{1}{n+1}$$

satisfies

$$X_n > 0$$

at every finite stage, yet

$$[X] = 0_{\mathbb{R}_{\mathrm{C}}}.$$

Example 56.21 (Exact zero at every stage)**.** *The constant family*

$$Z_n = 0$$

has a typed finite exact-zero report at every stage and also satisfies

$$[Z] = 0_{\mathbb{R}_{\mathrm{C}}}.$$

The two source families are not constructively identical:

$$X \neq_{\mathrm{C}} Z.$$

They become equal only after the completion quotient discards the distinction between their vanishing residual histories.

Theorem 56.22 (No nonzero rational enters the completed-zero class)**.** *For $q \in \mathbb{Q}_{\mathrm{C}}$,*

$$\iota(q) = 0_{\mathbb{R}_{\mathrm{C}}} \quad \Longleftrightarrow \quad q = 0.$$

Proof. This is injectivity of the rational embedding applied to q and 0. □

Principle 56.23 (Typed completed zero)**.** *A completed zero reports zero Cauchy distance from the constant-zero family. It does not assert that every representative has a finite exact zero, that every stage lacks positive occurrences, or that every zero representative shares one construction history.*

56.10 Information discarded by real completion

The quotient map κ identifies all Cauchy-equivalent approximation families. Consequently, a completed scalar generally does not retain:

(i) the source rule that generated the approximations;

(ii) the chosen rational representatives;

(iii) the base or digit grammar of a positional name;

(iv) the side or orientation of approach;

(v) monotonicity, alternation, or finite overshoot;

(vi) the rate at which resolutions are met;

(vii) the witness map $\rho \mapsto N_\rho$;

(viii) whether any finite stage equals the completed value;

(ix) whether the family eventually stabilizes;

(x) the nested interval, recurrence, geometric, or algebraic certificate used to prove Cauchy coherence;

(xi) or the source-level zeros and cancellations occurring along the way.

Example 56.24 (One completed rational, different approximation histories)**.** *The families*

$$A_n = 1, \qquad B_n = 1 - 10^{-(n+1)}, \qquad C_n = 1 + (-1)^n 2^{-(n+1)}$$

all represent the same completed real:

$$[A] = [B] = [C] = \iota(1).$$

The first is stationary, the second approaches from below, and the third alternates around the target. Completion preserves their common scalar and discards those stagewise distinctions unless they are carried in a richer report.

Theorem 56.25 (Completion-fiber preservation)**.** *A property P of rational Cauchy families descends to a property of completed real reports exactly when*

$$X \sim_{\mathrm{Cau}} Y \Longrightarrow P(X) = P(Y).$$

An operation descends exactly when it respects $\sim_{\mathrm{Cau}}$ in every input lane.

Proof. This is the projection-fiber factorization theorem of Section 54 applied to the quotient map κ. □

56.11 Rational completion theorem

Theorem 56.26 (Constructive rational completion)**.** *The carrier $\mathbb{R}_{\mathrm{C}}$ satisfies the following completion obligations:*

(i) every rational Cauchy family determines one completed report;

(ii) two families determine the same report exactly when their synchronized residual tends to zero;

(iii) the finite rational carrier embeds injectively by constant families;

(iv) rational convergence agrees with the embedding;

(v) nested rational intervals of vanishing width determine the same completed reports as Cauchy families;

(vi) addition, negation, subtraction, and multiplication are well defined on completion fibers;

(vii) and the completed-zero class is distinct from the requirement of finite exact zero.

The completion is therefore not an unexplained continuum inserted beneath finite arithmetic. It is a declared quotient of coherent finite rational reports, together with explicit preservation proofs for the operations admitted on the quotient.

57 Irrational reports and completed roots

The completion carrier contains reports not equal to any embedded rational. These reports must be distinguished from their construction rules, positional names, and algebraic predicates.

57.1 Rational and irrational completed reports

Definition 57.1 (Rational completed real)**.** *A completed real $\alpha \in \mathbb{R}_{\mathrm{C}}$ is* rational *when there exists $q \in \mathbb{Q}_{\mathrm{C}}$ such that*

$$\alpha = \iota(q).$$

Definition 57.2 (Irrational completed real)**.** *A completed real α is* irrational *when no finite rational report embeds to it:*

$$\forall q \in \mathbb{Q}_{\mathrm{C}}, \qquad \alpha \neq \iota(q).$$

Irrationality is therefore a statement about the completion fiber. It does not mean that the report lacks a finite rule, a finite symbol, a root predicate, or finite rational approximations.

Principle 57.3 (Finite description does not imply rational value)**.** *A finite construction rule may generate a Cauchy family whose completed report is irrational. Finiteness of the rule and rationality of the completed scalar are different properties.*

57.2 Root interval construction

Let $A \in \mathbb{Q}_{\mathrm{C}}$ be positive and let $k \geq 2$ be a finite natural exponent. Section 51 constructed exact rational roots and finite residual root shells. Completion now assigns a target when rational shells contract without exact rational closure.

Choose rational bounds

$$0 \leq L_0 < U_0$$

satisfying

$$L_0^k \le A \le U_0^k.$$

For example, one may take

$$L_0 = 0, \qquad U_0 = \max(1, A).$$

Given $[L_n, U_n]$, form the rational midpoint

$$M_n := \frac{L_n + U_n}{2}.$$

Define the next interval by the exact finite comparison

$$[L_{n+1}, U_{n+1}] = \begin{cases} [L_n, M_n], & A \le M_n^k, \\ [M_n, U_n], & M_n^k < A. \end{cases}$$

If $M_n^k = A$, either branch may be replaced by the constant exact interval $[M_n, M_n]$.

Theorem 57.4 (Root intervals are nested and contractive)**.** *The root interval rule satisfies*

$$L_n \le L_{n+1} \le U_{n+1} \le U_n,$$

$$L_n^k \le A \le U_n^k,$$

and

$$U_n - L_n = 2^{-n}(U_0 - L_0).$$

Hence its widths converge to zero.

Proof. Each step retains one half of the preceding interval, so nesting and the width formula are immediate by finite induction. The branch is chosen precisely to preserve the power enclosure. Since $2^{-n} \to 0$, the widths converge to zero. □

Definition 57.5 (Completed nonnegative root report)**.** *The completed kth-root report generated by the interval rule is*

$$\sqrt[k]{A}_{\mathrm{C}} := [(L_n)] = [(U_n)] \in \mathbb{R}_{\mathrm{C}}.$$

The notation $\sqrt[k]{A}_{\mathrm{C}}$ abbreviates the completed interval fiber. It is not a claim that a radical symbol existed before the interval construction.

57.3 Closure of the root relation

Theorem 57.6 (Completed root relation)**.** *For the completed root report $\alpha = \sqrt[k]{A}_{\mathrm{C}}$,*

$$\alpha^k = \iota(A).$$

Proof. Let M be a positive rational bound for all endpoints; one may take $M = U_0 + 1$. For nonnegative $L_n \le U_n$,

$$U_n^k - L_n^k = (U_n - L_n) \sum_{j=0}^{k-1} U_n^{k-1-j} L_n^j.$$

Every term in the sum is at most M^{k-1}, so

$$0 \le U_n^k - L_n^k \le k M^{k-1}(U_n - L_n) \longrightarrow_{\mathrm{C}} 0.$$

Because

$$L_n^k \le A \le U_n^k,$$

we have

$$|L_n^k - A| \le U_n^k - L_n^k \longrightarrow_{\mathrm{C}} 0.$$

Therefore the product class $[(L_n^k)]$ equals the constant class $\iota(A)$, which is exactly

$$\alpha^k = \iota(A).$$

□

At every finite stage, the lower endpoint may retain a nonzero root residual

$$A - L_n^k > 0.$$

The completed root relation is exact because that residual family belongs to the completed-zero class, not because a positive finite residual becomes a finite exact zero.

57.4 Exact rational roots versus irrational completed roots

Theorem 57.7 (Exact-root collapse)**.** *If A has an exact nonnegative rational kth root q, so that*

$$q^k = A,$$

then every contracting root interval specification retaining q represents

$$\sqrt[k]{A}_{\mathrm{C}} = \iota(q).$$

Proof. For every stage,

$$L_n \le q \le U_n.$$

Interior choice invariance from Section 56 identifies the endpoint class with the constant family q. □

Corollary 57.8 (Non-rational root reports)**.** *If no $q \in \mathbb{Q}_{\mathrm{C}}$ satisfies $q^k = A$, then the completed root report generated above is irrational.*

Proof. If the completed root were $\iota(q)$ for some rational q, the well-defined power operation would give

$$\iota(q^k) = \iota(A).$$

Injectivity of ι would imply $q^k = A$, contrary to hypothesis. □

57.5 The completed report represented by $\sqrt{2}$

Begin with

$$I_0 = [1, 2].$$

The first midpoint is $3/2$, whose square is $9/4 > 2$, so

$$I_1 = \left[1, \frac{3}{2}\right].$$

The next midpoint is $5/4$, whose square is $25/16 < 2$, so

$$I_2 = \left[\frac{5}{4}, \frac{3}{2}\right].$$

The next midpoint is $11/8$, whose square is $121/64 < 2$, so

$$I_3 = \left[\frac{11}{8}, \frac{3}{2}\right].$$

The rule continues by exact rational squaring and comparison. The interval widths halve at every stage.

Definition 57.9 (Constructive square-root-of-two report)**.** *Let*

$$\sqrt{2}_{\mathrm{C}} := [(L_n)] = [(U_n)]$$

for the preceding bisection interval family.

The completed root theorem gives

$$(\sqrt{2}_{\mathrm{C}})^2 = \iota(2).$$

Theorem 57.10 (Irrationality of the completed square root of two)**.** *The report $\sqrt{2}_{\mathrm{C}}$ is irrational.*

Proof. Assume

$$\sqrt{2}_{\mathrm{C}} = \iota(p/q)$$

for positive integers p, q with their common replication factors cancelled. Squaring and using injectivity of the rational embedding gives

$$p^2 = 2q^2.$$

Thus p^2 is even. A finite parity decomposition shows that an odd integer has an odd square, so p must be even. Write $p = 2r$. Substitution gives

$$4r^2 = 2q^2, \qquad q^2 = 2r^2.$$

Hence q is also even. Both numerator and denominator therefore share a factor of two, contradicting the declared reduced construction of p/q. No rational report embeds to $\sqrt{2}_{\mathrm{C}}$. □

The contradiction concerns finite numerator-denominator structure. It does not depend on reading an infinite decimal expansion.

57.6 Four distinct reports associated with an irrational root

For a completed root such as $\sqrt{2}_{\mathrm{C}}$, at least four report layers must be distinguished.

(i) **Construction rule.** The finite bisection rule that maps one rational interval to the next.

(ii) **Positional expansion.** A base-dependent sequence of digit-prefix reports whose cylinder intervals retain the completed value.

(iii) **Root-defining relation.** The completed algebraic predicate

$$x^2 = 2.$$

(iv) **Completed scalar report.** The Cauchy equivalence class

$$\sqrt{2}_{\mathrm{C}} \in \mathbb{R}_{\mathrm{C}}.$$

These layers are related but not identical. The bisection rule is not a decimal string. A decimal string is not the equation $x^2 = 2$. The equation may have multiple roots after signed orientation is admitted. The completed scalar suppresses many source rules capable of constructing the same root.

Principle 57.11 (Root-predicate preservation)**.** *The predicate $P(X) := [X]^k = \iota(A)$ descends to completion because multiplication is well defined on Cauchy fibers. A stagewise predicate such as "the nth approximation was produced by bisection" does not descend unless separately retained.*

57.7 Positional names for completed reals

Fix a finite base $b \geq 2$. A length-m radix prefix determines a rational cylinder interval of width b^{-m} consisting of all completed reports sharing that prefix under the declared digit convention.

Definition 57.12 (Completed positional name)**.** *A* base-b positional name *for a completed real α is a coherent family of finite digit prefixes*

$$D_1 \prec D_2 \prec D_3 \prec \cdots$$

whose associated rational cylinder intervals are nested, retain α, and have widths b^{-m}.

The positional name is therefore another nested interval specification. It reports the same completed scalar only after the interval-to-Cauchy equivalence of Section 56 is applied.

Theorem 57.13 (Eventually periodic positional names are rational)**.** *If a base-b positional name is eventually periodic, then its completed value is rational.*

Proof. Separate the finite nonperiodic prefix from one repeating block. The repeating tail is a finite geometric ratio whose common factor is b^{-r}, where r is the block length. Finite ratio arithmetic gives a rational closed form. Adding the finite prefix preserves rationality. □

Theorem 57.14 (Rational positional names eventually recur)**.** *Every rational completed real has an eventually terminating or eventually recurring base-b positional name.*

Proof. This is the finite remainder recurrence theorem of Section 49 applied to a finite rational representative. Completion identifies the generated prefix family with the embedded rational value. □

Corollary 57.15 (Irrational positional nonperiodicity)**.** *No base-b positional name of an irrational completed real is eventually periodic.*

This criterion concerns completed positional names, not the finiteness of the rule generating them. A finite algorithm may produce a nonperiodic digit family.

57.8 Dual positional reports and normalization

Some rational boundary values admit two positional names, such as a terminating name and a name with an indefinitely repeated maximal digit. Their finite prefixes differ, but their cylinder intervals determine the same completion class.

Example 57.16 (Two positional names, one completed scalar)**.** *Under the limit-value interpretation,*

$$1.000\ldots_{10} =_{\mathbb{R}_{\mathrm{C}}} 0.999\ldots_{10}.$$

The left rule stabilizes after the radix boundary. The right rule retains a positive residual at every finite prefix. They are distinct rule reports and equal completed scalars.

A canonical positional convention may prohibit an indefinitely repeated maximal digit in order to choose one representative. Such normalization selects a preferred name; it does not prove that the discarded name never existed.

57.9 Completed residual roots

Let L_n be the lower endpoint family for a completed kth root of A. Define the finite root residual

$$R_n := A - L_n^k \geq 0.$$

Theorem 57.17 (Root residual completion)**.** *The residual family satisfies*

$$[R] = 0_{\mathbb{R}_{\mathrm{C}}}.$$

If no finite endpoint is an exact root, then

$$R_n > 0$$

for every finite n even though its completed report is zero.

Proof. The completed root theorem proves $L_n^k \sim_{\mathrm{Cau}} A$, hence $R_n \to 0$. The second statement is the assumed failure of exact finite closure. □

This is the root analogue of the decimal-nine residual. Completion closes a vanishing residual family without retroactively replacing each finite residual by exact zero.

57.10 Completed-root preservation theorem

Theorem 57.18 (Completed-root report preservation)**.** *A completed root report retains:*

(i) its completed scalar class;

(ii) every algebraic predicate proved invariant under Cauchy equivalence, including the completed power relation;

(iii) and any separately attached source certificate.

It does not automatically retain:

(iv) the bisection, Newton, positional, geometric, or other approximation rule;

(v) the finite side of approach;

(vi) the exact residual at each stage;

(vii) the base of a positional name;

(viii) or the finite stage at which a requested precision was reached.

The distinction among rule, name, predicate, and scalar will be essential later when analytic functions receive the same scalar output from different source constructions.

58 Real order, bounds, and completeness

Completion creates scalar fibers. It does not by itself justify comparing those fibers. Signed order must be shown independent of representatives, and the different meanings of completeness must be stated rather than conflated.

58.1 Approximate non-strict order

Let $X = (X_n)$ and $Y = (Y_n)$ be rational Cauchy representatives.

Definition 58.1 (Completed non-strict order)**.** *Define*

$$[X] \leq_{\mathbb{R}_C} [Y]$$

when, for every positive rational resolution ρ, there exists a finite N_ρ such that

$$n \geq N_\rho \Longrightarrow X_n \leq Y_n + \rho.$$

This definition allows finite approximation noise. It does not require the chosen representatives to satisfy $X_n \leq Y_n$ at every stage.

Theorem 58.2 (Order is representative-independent)**.** *If $X \sim_{\mathrm{Cau}} X'$ and $Y \sim_{\mathrm{Cau}} Y'$, then*

$$[X] \leq [Y] \quad \Longleftrightarrow \quad [X'] \leq [Y'].$$

Proof. Assume $[X] \leq [Y]$ and request positive resolution ρ. On a common tail, require

$$|X_n - X'_n| < \rho/3, \qquad |Y_n - Y'_n| < \rho/3, \qquad X_n \leq Y_n + \rho/3.$$

Then

$$X'_n \leq X_n + \rho/3 \leq Y_n + 2\rho/3 \leq Y'_n + \rho.$$

The reverse implication is symmetric. □

58.2 Strict separation order

Definition 58.3 (Completed strict order)**.** *Define*

$$[X] <_{\mathbb{R}_C} [Y]$$

when there exist a positive rational separation δ and a finite index N such that

$$n \geq N \Longrightarrow X_n + \delta \leq Y_n.$$

Strict order therefore requires an eventually retained positive gap. A finite crossing or occasional equality does not destroy strict order when a later uniform separation exists.

Theorem 58.4 (Strict order is representative-independent)**.** *The relation $<_{\mathbb{R}_C}$ is constant on Cauchy equivalence classes.*

Proof. If $X_n + \delta \le Y_n$ eventually, choose a common tail on which

$$|X_n - X'_n| < \delta/3, \qquad |Y_n - Y'_n| < \delta/3.$$

Then

$$X'_n + \delta/3 \le X_n + 2\delta/3 \le Y_n - \delta/3 \le Y'_n.$$

Thus the replacement representatives retain the positive gap $\delta/3$. □

Definition 58.5 (Positive and negative completed reports)**.** *A completed real A is positive when*

$$0_{\mathbb{R}_C} < A,$$

and negative when

$$A < 0_{\mathbb{R}_C}.$$

This extends signed orientation from finite integer and rational reports to completed fibers.

58.3 Order laws

Theorem 58.6 (Completed order is a partial order)**.** *The relation $\le_{\mathbb{R}_C}$ is reflexive, transitive, and antisymmetric.*

Proof. Reflexivity is immediate. For transitivity, split an arbitrary requested resolution into two positive parts and combine the eventual inequalities. For antisymmetry, suppose $[X] \le [Y]$ and $[Y] \le [X]$. Given positive ρ, on a common tail one has

$$X_n \le Y_n + \rho \quad \text{and} \quad Y_n \le X_n + \rho.$$

Hence

$$|X_n - Y_n| \le \rho.$$

Using $\rho/2$ in the hypotheses gives strict containment below any requested ρ, so $X \sim_{\mathrm{Cau}} Y$ and $[X] = [Y]$. □

Theorem 58.7 (Compatibility with addition and negation)**.** *For completed reals A, B, C:*

$$A \le B \Longrightarrow A + C \le B + C,$$

$$A < B \Longrightarrow A + C < B + C,$$

and

$$A \le B \Longrightarrow -B \le -A.$$

Proof. All three statements follow from the corresponding finite rational inequalities applied on the eventual tails. Addition preserves the same residual gap, while negation reverses its orientation. □

Theorem 58.8 (Positive-product compatibility)**.** *If*

$$0 \le A, \qquad 0 \le B,$$

then

$$0 \le AB.$$

If in addition $0 < A$ *and* $0 < B$*, then* $0 < AB$*.*

Proof. For non-strict positivity, any finite negative excursions of chosen representatives are dominated by arbitrarily small positive resolutions; replacing them by equivalent nonnegative tail representatives, or applying the resolution definition directly, gives $0 \le AB$. For strict positivity, choose eventual positive gaps α and β so that representatives satisfy $A_n \ge \alpha$ and $B_n \ge \beta$. Then

$$A_n B_n \ge \alpha\beta > 0$$

on that tail. □

58.4 Linearity and its logical dependency

For arbitrary completed reports, a proof of

$$A \le B \quad \text{or} \quad B \le A$$

requires a decision about the completed sign of $B - A$. That decision is not contained in any fixed finite prefix.

Principle 58.9 (Located comparison principle)**.** *A completed pair* (A, B) *is* located *when, for every positive rational* ρ*, one can construct at least one of the reports*

$$A < B + \rho \qquad \text{or} \qquad B < A + \rho.$$

A family is located when such comparisons can be produced uniformly for its relevant members and rational bounds.

Principle 58.10 (Declared linear-order principle)**.** *If the logic admits the completed sign decision needed for every pair, then* $\le_{\mathbb{R}_C}$ *is a linear order. Under a stricter constructive reading, the preceding partial order and strict-separation relation remain valid, while trichotomy is asserted only for located or otherwise decidable pairs.*

This dependency must be visible. A geometric drawing of a line cannot supply a comparison decision that the report grammar has not constructed.

58.5 Completed intervals

Definition 58.11 (Closed completed interval)**.** *For* $A \le B$*, define*

$$[A, B]_{\mathbb{R}_C} := \{X \in \mathbb{R}_C : A \le X \le B\}.$$

Definition 58.12 (Open completed interval)**.** *For* $A < B$*, define*

$$(A, B)_{\mathbb{R}_C} := \{X \in \mathbb{R}_C : A < X < B\}.$$

The interval is a relational report between completed endpoints. It is not yet a physical segment or a claim that uncountably many objects have been physically placed between them.

Theorem 58.13 (Rational density)**.** *If $A < B$, then there exists $q \in \mathbb{Q}_{\mathrm{C}}$ such that*

$$A < \iota(q) < B.$$

Proof. Choose representatives and a positive rational gap δ with

$$A_n + \delta \leq B_n$$

eventually. Select one sufficiently late stage and choose the rational midpoint

$$q := \frac{A_n + B_n}{2}.$$

After making the representative errors smaller than $\delta/4$, the constant class $\iota(q)$ remains separated from both completed endpoints by a positive rational gap. □

Theorem 58.14 (Archimedean bound)**.** *For every $A \in \mathbb{R}_{\mathrm{C}}$, there exists a natural integer report N such that*

$$A < \iota(N).$$

Likewise, some negative integer lies below A.

Proof. Choose a rational Cauchy representative of A. Cauchy boundedness supplies a rational magnitude bound M. Choose a natural integer $N > M + 1$. Then the constant report N is eventually separated above every representative term. Negation gives the lower bound. □

58.6 Upper and lower bounds

Definition 58.15 (Upper bound)**.** *Let S be a report family of completed reals. A completed real U is an* upper bound *of S when*

$$\forall X \in S, \qquad X \leq U.$$

Definition 58.16 (Lower bound)**.** *A completed real L is a* lower bound *of S when*

$$\forall X \in S, \qquad L \leq X.$$

Definition 58.17 (Least upper bound)**.** *A completed real S^* is the* least upper bound *or* supremum *of S when:*

(i) S^* is an upper bound of S;

(ii) every upper bound U of S satisfies $S^* \leq U$.

Write

$$S^* = \sup S.$$

Definition 58.18 (Greatest lower bound)**.** *A completed real S_* is the* greatest lower bound *or* infimum *of S when:*

(i) S_* is a lower bound of S;

(ii) every lower bound L of S satisfies $L \leq S_*$.

Write

$$S_* = \inf S.$$

Supremum and infimum are order reports. They do not require that the bound itself be a member of the family.

58.7 Nested-interval completeness

Theorem 58.19 (Nested completed interval theorem)**.** *Let*

$$I_n = [A_n, B_n]_{\mathbb{R}_C}$$

be nonempty closed completed intervals satisfying

$$I_{n+1} \subseteq I_n$$

and

$$B_n - A_n \longrightarrow 0.$$

Then there exists exactly one completed real L lying in every I_n.

Proof. Choose rational approximations a_n, b_n to A_n, B_n at resolution smaller than $2^{-(n+3)}$, while preserving a nested rational enclosure by widening each endpoint by that resolution and intersecting with the preceding enclosure. This produces a nested rational interval specification of width tending to zero. Section 56 assigns it a completed real L. The approximation widths show $A_n \le L \le B_n$ for every fixed n. If L' also lies in every interval, then

$$|L - L'| \le B_n - A_n$$

for every n, so the residual is below every positive resolution and $L = L'$. □

The theorem is the completed form of nested whole bounds. It supplies one scalar fiber, not a finite stage at which the interval literally has zero width.

58.8 Sequential Cauchy completeness

Definition 58.20 (Cauchy family of completed reals)**.** *A family (A_n) in $\mathbb{R}_C$ is Cauchy when, for every positive rational resolution ρ, there exists N_ρ such that*

$$m, n \ge N_\rho \Longrightarrow |A_m - A_n| < \rho.$$

Here $|A_m - A_n| < \rho$ abbreviates strict separation of the residual magnitude below the embedded positive rational ρ.

Theorem 58.21 (Sequential completeness)**.** *Every Cauchy family of completed reals converges to a member of $\mathbb{R}_C$.*

Proof. Choose an increasing index sequence N_k such that

$$m, n \ge N_k \Longrightarrow |A_m - A_n| < 2^{-(k+2)}.$$

For each k, choose a rational representative q_k of A_{N_k} within $2^{-(k+2)}$ of that completed report. Such a rational stage exists because every completed real is represented by a rational Cauchy family.

For $j \ge k$,

$$|q_j - q_k| \le |q_j - A_{N_j}| + |A_{N_j} - A_{N_k}| + |A_{N_k} - q_k| < 2^{-k}.$$

Thus (q_k) is a rational Cauchy family and defines

$$L := [(q_k)] \in \mathbb{R}_C.$$

The same estimates show $A_n \to L$. □

Sequential completeness is built into the Cauchy completion. It is not an additional picture of a filled coordinate line.

58.9 Located least-upper-bound construction

Full least-upper-bound claims require enough information to decide how a bounded family sits relative to rational cuts.

Definition 58.22 (Supremum-located family)**.** *An inhabited family $S \subseteq \mathbb{R}_{\mathrm{C}}$ is* supremum-located *when, for every rational pair $a < b$, one can construct at least one of the following reports:*

(i) b is an upper bound of S;

(ii) there exists $X \in S$ with $a < X$.

This form is deliberately weaker than deciding whether a single rational is exactly an upper bound. The positive gap $a < b$ leaves room for finite approximation.

Theorem 58.23 (Least upper bound for located bounded families)**.** *Every inhabited, supremum-located family of completed reals that has an upper bound has a least upper bound in $\mathbb{R}_{\mathrm{C}}$.*

Proof. By the Archimedean theorem, choose rational bounds $L_0 < U_0$ such that U_0 is an upper bound and some member of S lies above L_0.

At stage n, let

$$M_n := \frac{L_n + U_n}{2}.$$

Apply locatedness with a narrow rational buffer around M_n. If the upper test is returned, choose the upper half endpoint so that the new U_{n+1} remains an upper bound. If the witness test is returned, choose L_{n+1} below the witnessed member, certifying that L_{n+1} is not an upper bound. The buffer can be chosen so that the interval width is at most a fixed fraction smaller than the preceding width.

This produces nested rational intervals

$$[L_n, U_n]$$

of vanishing width, with every U_n an upper bound and every L_n certified not to be an upper bound. Let

$$\alpha := \mathrm{Real}([L_n, U_n]).$$

To show that α is an upper bound, suppose some $X \in S$ satisfied $\alpha < X$. Rational separation would eventually place X above one of the upper endpoints U_n, contradicting that U_n is an upper bound.

To show leastness, let $Y < \alpha$. Rational separation and vanishing interval width eventually give $Y < L_n$. Since L_n is not an upper bound, some $X \in S$ satisfies $L_n < X$, and hence $Y < X$. Therefore Y is not an upper bound. Every upper bound must satisfy $\alpha \leq Y$. Thus $\alpha = \sup S$. □

Corollary 58.24 (Greatest lower bound for located bounded families)**.** *If S is inhabited, suitably infimum-located, and bounded below, then*

$$\inf S = -\sup(-S),$$

where

$$-S := \{-X : X \in S\}.$$

58.10 Completeness is a typed property

The word *complete* now has several nonidentical meanings.

Definition 58.25 (Sequential completeness)**.** *A carrier is sequentially complete when every Cauchy sequence in the carrier converges to a member of the carrier.*

Definition 58.26 (Nested-interval completeness)**.** *A carrier is nested-interval complete when every nested family of nonempty closed intervals whose widths tend to zero has exactly one common member.*

Definition 58.27 (Located order completeness)**.** *A carrier is located-order complete when every inhabited located family that is bounded above has a supremum, and dually for infima.*

Definition 58.28 (Full order completeness)**.** *A carrier is fully order complete when every inhabited family that is bounded above has a supremum, without an additional locatedness or decision certificate.*

Theorem 58.29 (Established completeness of the constructed carrier)**.** *The carrier $\mathbb{R}_{\mathrm{C}}$ constructed in Section 56 is:*

(i) sequentially complete;

(ii) nested-interval complete;

(iii) and order complete for inhabited located bounded families.

Principle 58.30 (Full order-completeness declaration)**.** *Full order completeness for arbitrary bounded report families requires an additional comprehension, choice, or sign-location principle strong enough to perform the needed cut decisions. If that principle is adopted, then every nonempty upper-bounded family has a supremum and every nonempty lower-bounded family has an infimum. If it is not adopted, no proof may silently use the full least-upper-bound property outside the located domain established above.*

This is not a defect hidden behind terminology. It is the explicit dependency separating Cauchy completion from unrestricted order completion.

58.11 Completed reciprocal for separated nonzero reports

The order now permits the missing reciprocal construction.

Definition 58.31 (Separated nonzero completed real)**.** *A completed real $A = [(A_n)]$ is* separated from zero *when there exists a positive rational δ and a finite N such that*

$$n \geq N \Longrightarrow |A_n| \geq \delta.$$

Every strictly positive or strictly negative completed real is separated from zero.

Theorem 58.32 (Completed reciprocal)**.** *If A is separated from zero, then the tail family*

$$A_n^{-1}$$

is Cauchy, its completion class is independent of representatives and finite initial choices, and

$$A \cdot A^{-1} = 1.$$

Proof. For tail indices,

$$\left|\frac{1}{A_m}-\frac{1}{A_n}\right| = \frac{|A_m - A_n|}{|A_m A_n|} \leq \frac{|A_m - A_n|}{\delta^2}.$$

Cauchy coherence of A therefore gives Cauchy coherence of the reciprocal family. Equivalent representatives remain close enough to preserve a smaller positive lower bound, and the same estimate proves representative independence. Finally, every tail stage satisfies $A_n A_n^{-1} = 1$, so the product class is the embedded unit. □

Principle 58.33 (Nonzero versus apart from zero)**.** *Under a fully linear classical order, $A \neq 0$ implies either $A > 0$ or $A < 0$, hence separation from zero. Under a stricter constructive reading, mere negated equality need not supply a positive separation witness. Reciprocal construction therefore requires apartness from zero, not merely the absence of a proof of equality.*

58.12 Part XIII completion summary

The construction developed in Sections 55–58 is

$$\begin{aligned}\text{finite signed ratios} &\longrightarrow \text{finite residual resolutions} \longrightarrow \text{Cauchy tail coherence}\\ &\longrightarrow \mathrm{Cau}(\mathbb{Q}_{\mathrm{C}})/\!\sim_{\mathrm{Cau}} \longrightarrow \mathbb{R}_{\mathrm{C}}\\ &\longrightarrow \text{completed arithmetic, roots, order, bounds, and typed completeness.}\end{aligned}$$

The essential conclusions are:

(1) a real report is a Cauchy-equivalence class of finite rational approximation families;

(2) finite rationals embed by constant families and retain their distinctions;

(3) nested rational intervals and Cauchy families yield the same completed carrier;

(4) completed equality forgets rule, rate, side, base, and finite-stage history unless separately retained;

(5) completed zero need not contain a finite exact-zero stage;

(6) irrational reports are completed values outside the rational embedding, not objects without finite descriptions;

(7) a completed root closes a vanishing residual family without turning any positive finite residual into exact zero;

(8) positional expansion, root predicate, construction rule, and completed scalar are different report layers;

(9) signed order is defined by eventual finite-resolution comparison and must be proved independent of representatives;

(10) the Cauchy completion is sequentially and nested-interval complete;

(11) least upper bounds are constructed for located bounded families;

(12) unrestricted order completeness is an additional declared logical commitment rather than a hidden consequence of the word “real”;

(13) and completed reciprocals require a positive separation from zero.

The next part should return to geometry and measure. It must construct boundary, closure, interior, positive measure, and simplex depth before coordinate projection is allowed to report those structures.

Part XIV
Geometry and Measure

59 Boundary, closure, and positive measure

The preceding parts constructed finite reports, signed orientation, completed real magnitudes, and the equality relations needed to distinguish a construction from its coordinate value. Geometry may now be extended without treating coordinates as primitive. The order of dependence is

$$\begin{aligned}\text{distinguishable placements} &\longrightarrow \text{incidence} \longrightarrow \text{oriented boundary} \longrightarrow \text{closure}\\ &\longrightarrow \text{interior/exterior separation} \longrightarrow \text{positive measure.}\end{aligned}$$

A coordinate formula may report the resulting closure, but it does not create the closure.

59.1 Incidence reports

Definition 59.1 (Geometric incidence report)**.** *A finite geometric incidence report is a tuple*

$$\mathcal{K} = (V, E, F_2, F_3, \ldots, \iota, \Omega, \gamma),$$

where:

(i) V *is a finite family of distinguishable placements;*

(ii) E *is a finite family of line relations between placements;*

(iii) F_k *is a finite family of declared* k*-dimensional cells when such cells have been constructed;*

(iv) ι *records which lower-dimensional cells occur in the boundary of which higher-dimensional cells;*

(v) Ω *records orientation;*

(vi) γ *records source and construction history.*

An incidence report is pre-coordinate. It states which placements and relations participate in a construction before assigning numerical coordinates to them.

Definition 59.2 (Oriented edge)**.** *For distinct placements* p_i *and* p_j*, an oriented edge is the directed incidence*

$$[p_i, p_j].$$

Its reversed orientation is

$$-[p_i, p_j] := [p_j, p_i].$$

The two orientations have the same unoriented line report but opposite boundary direction.

Definition 59.3 (Boundary of an oriented edge)**.** *The boundary of an oriented edge is the signed endpoint report*

$$\partial_1[p_i, p_j] := [p_j] - [p_i].$$

The positive endpoint reports arrival and the negative endpoint reports departure. These are orientation lanes, not negative placements.

Definition 59.4 (Finite oriented chain)**.** *A finite oriented k-chain is a source-preserving signed join of oriented k-cells,*

$$C_k = \sum_{r=1}^{m} \varepsilon_r \sigma_r, \qquad \varepsilon_r \in \{+1, -1\}.$$

The sum notation abbreviates a retained ledger of whole cells and their orientations.

The chain notation does not divide or blend cells. Each σ_r remains a whole indexed occurrence. Opposite signs indicate opposite orientation or cancellation obligations.

59.2 Open and closed incidence

Definition 59.5 (Open incidence chain)**.** *A finite oriented k-chain C_k is* open *when its boundary report contains at least one unmatched positive or negative $(k-1)$-cell:*

$$\partial_k C_k \neq 0_{\partial C_k}.$$

Definition 59.6 (Closed incidence chain)**.** *A finite oriented k-chain C_k is* closed *when every boundary occurrence is paired with an occurrence of opposite orientation and the resulting typed boundary report is zero:*

$$\partial_k C_k = 0_{\partial C_k}.$$

The closure zero is constructive. It certifies that every declared boundary obligation closes; it is not literal absence of the chain.

Example 59.7 (Open and closed one-dimensional chains)**.** *For*

$$C_1 = [p_0, p_1] + [p_1, p_2],$$

we have

$$\partial_1 C_1 = ([p_1] - [p_0]) + ([p_2] - [p_1]) = [p_2] - [p_0].$$

The intermediate endpoint cancels, but the first and last endpoints remain. The chain is open.

For

$$Z_1 = [p_0, p_1] + [p_1, p_2] + [p_2, p_0],$$

all endpoint obligations cancel:

$$\partial_1 Z_1 = 0_{\partial Z_1}.$$

The cycle is closed.

Definition 59.8 (Oriented simplex boundary)**.** *For an oriented k-simplex*

$$\sigma = [p_0, p_1, \ldots, p_k],$$

define

$$\partial_k \sigma = \sum_{j=0}^{k} (-1)^j [p_0, \ldots, \widehat{p_j}, \ldots, p_k],$$

where the hat indicates omission of that placement.

This is later notation for the alternating orientation ledger of the simplex faces.

Theorem 59.9 (Boundary-of-boundary closure)**.** *For every oriented finite simplex σ,*

$$\partial_{k-1}(\partial_k \sigma) = 0.$$

The same holds for every finite oriented simplicial chain by source-preserving additivity.

Proof. Every $(k-2)$-face of σ is obtained by omitting two placements. It occurs exactly twice in $\partial_{k-1}(\partial_k \sigma)$, once for each order in which those placements are omitted. The two occurrences receive opposite orientation signs. They therefore cancel as whole, source-matched boundary obligations. No positive boundary occurrence survives. □

Principle 59.10 (Closure is not mere return of a scalar)**.** *A scalar endpoint equality may report that a path begins and ends at the same coordinate. Constructive closure additionally requires that the declared incidence, orientation, adjacency, and boundary obligations close. Equal endpoint coordinates alone do not certify a simple closed boundary.*

59.3 Boundary, interior, and exterior

Definition 59.11 (Boundary carrier)**.** *For a completed finite k-cell K, its boundary carrier is the retained $(k-1)$-chain*

$$\partial K.$$

A lower-dimensional cell belongs to the exposed boundary when it is incident to exactly one selected k-cell after opposite internal incidences have cancelled.

Thus, in a triangle complex, an edge shared by two oppositely oriented triangles is internal and cancels from the exposed boundary. An edge incident to only one triangle remains exposed.

Theorem 59.12 (Finite incidence identity for triangle complexes)**.** *Let a finite nonoverlapping triangle complex contain N complete triangles, I complete shared edges, and P complete exposed edges. Then*

$$3N = 2I + P.$$

Proof. Count triangle-edge incidences. Every triangle contributes three incidences, giving $3N$. Every shared edge is incident to two triangles and contributes two incidences. Every exposed edge is incident to one triangle and contributes one incidence. Hence $3N = 2I + P$. □

This identity retains a whole-cell construction ledger. No area fraction is required.

Definition 59.13 (Exterior reference)**.** *An exterior reference for a finite embedded closure is a declared placement or region known not to lie in the bounded cell ledger.*

Definition 59.14 (Exterior accessibility)**.** *A placement is* exterior-accessible *when it can be joined to the exterior reference by an admissible path that does not cross the boundary carrier.*

Definition 59.15 (Constructive interior)**.** *A placement or admitted cell is in the* constructive interior *of a closed boundary when it is not part of the exposed boundary and every admissible path to the exterior reference crosses that boundary.*

Definition 59.16 (Constructive exterior)**.** *The* constructive exterior *consists of placements exterior-accessible without crossing the boundary, together with the declared exterior reference class.*

These definitions are relational. Coordinates may later locate an interior point, but the interior/exterior distinction is produced by separation under the boundary relation.

Principle 59.17 (Interior requires closure)**.** *An open boundary does not, by itself, separate a finite interior from an exterior. A coordinate drawing may visually suggest a region, but constructive interior requires a completed separation certificate.*

59.4 Dimensional contribution and constructive measure

Definition 59.18 (Dimension-typed contribution)**.** *A report X has positive constructive contribution in dimension k when it contains a completed k-cell with positive declared k-measure:*

$$\mu_k(X) > 0.$$

A report may have positive j-measure and zero k-measure for $j < k$. A line may have positive length but zero area; a triangle boundary has positive edge length but its boundary alone has zero area.

Definition 59.19 (Coordinate measure)**.** *A coordinate measure assigns a completed real magnitude to a geometric report after a coordinate frame, unit, and metric grammar have been declared.*

Definition 59.20 (Constructive measure)**.** *A constructive measure records which completed dimensional cells contribute the positive measure before coordinate normalization. It retains at least cell identity, dimension, adjacency, orientation, boundary, and overlap rules.*

Principle 59.21 (Zero-measure aggregation)**.** *Under ordinary finite or countable subadditivity, a finite or countable union of objects each having zero k-measure still has zero k-measure. Therefore positive k-measure cannot be attributed to ordinary addition of literal zero-k-measure contributions alone. A completed continuum, a new cell type, or another declared extension must supply the positive-measure structure.*

This principle does not reject measure theory or completion. It identifies the extension being used. The boundary of a triangle may have zero area while the completed triangular cell has positive area; the positive cell is not obtained by merely adding the zero-area vertices.

Definition 59.22 (Positive finite vertex-defined primitive)**.** *A positive finite vertex-defined k-primitive is a completed closed k-cell determined by finitely many placements and boundary relations, having positive k-measure, and minimal under the declared vertex-count comparison.*

59.5 Minimum positive area closure

Lemma 59.23 (One and two placements do not enclose area)**.** *One placement supplies no edge. Two distinct placements supply one undirected line relation. Neither construction closes a two-dimensional boundary.*

Lemma 59.24 (Three non-collinear placements suffice)**.** *Let p_0, p_1, p_2 be three placements not lying on one line relation. The three edges*

$$[p_0, p_1], \qquad [p_1, p_2], \qquad [p_2, p_0]$$

form a closed oriented boundary and distinguish an interior from an exterior. The resulting triangular cell has positive area under every faithful Euclidean coordinate report.

Proof. The boundary closes by endpoint cancellation. Non-collinearity prevents the three edges from collapsing into a one-dimensional carrier. A faithful coordinate realization produces two linearly independent displacement reports, and its signed two-lane closure is nonzero. Hence the enclosed two-dimensional measure is positive. □

Theorem 59.25 (Minimum finite positive-area primitive)**.** *Under the finite vertex-defined closure conditions above, the minimum primitive capable of positive area is a triangle.*

Proof. Fewer than three placements cannot close a two-dimensional boundary. Three non-collinear placements do close such a boundary and produce positive area. Therefore the minimum vertex count is three, and the corresponding primitive is a triangle. □

Definition 59.26 (Balanced minimum area primitive)**.** *A minimum area primitive is balanced when no vertex or edge is privileged by the declared length and incidence relations. The balanced minimum area primitive is therefore the regular, or equilateral, triangle.*

Principle 59.27 (The triangle is whole)**.** *The triangle is one complete minimum area closure. The coordinate formula*

$$A_{\triangle} = \frac{1}{2}bh$$

reports its area relative to a chosen parallelogram or rectangular carrier. The factor $1/2$ *does not make the triangle half of a primitive area object.*

A square or rectangle is a useful coordinate carrier, but it is composite relative to triangular closure: a diagonal decomposes it into two triangular cells.

59.6 Minimum positive volume closure

Lemma 59.28 (Three placements are volume-insufficient)**.** *Any three placements lie in a single two-dimensional affine carrier. Their closure may enclose area, but it cannot enclose positive three-dimensional volume.*

Lemma 59.29 (Four non-coplanar placements suffice)**.** *Let* p_0, p_1, p_2, p_3 *be four placements not contained in one plane relation. The four triangular faces*

$$[p_1, p_2, p_3], \quad -[p_0, p_2, p_3], \quad [p_0, p_1, p_3], \quad -[p_0, p_1, p_2]$$

form the closed boundary of a tetrahedral cell. A faithful Euclidean coordinate report assigns it positive volume.

Proof. Every edge belongs to two faces with opposite induced orientation, so the boundary of the face ledger closes. Non-coplanarity gives three independent displacement reports from one selected vertex. Their signed three-lane closure is nonzero, so the coordinate volume report is positive. □

Theorem 59.30 (Minimum finite positive-volume primitive)**.** *Under the finite vertex-defined closure conditions above, the minimum primitive capable of positive volume is a tetrahedron.*

Proof. Fewer than four placements cannot escape a two-dimensional affine carrier and therefore cannot enclose positive volume. Four non-coplanar placements close a tetrahedral boundary and do enclose positive volume. Hence the minimum is the tetrahedron. □

Definition 59.31 (Balanced minimum volume primitive)**.** *A minimum volume primitive is balanced when no vertex, edge, or face is privileged by the declared symmetry relations. The balanced minimum volume primitive is the regular tetrahedron.*

A cube is a useful orthogonal coordinate cell, but it is composite relative to tetrahedral closure.

59.7 The general simplex theorem

Definition 59.32 (Affine independence as a constructive report)**.** *Placements $p_0, \dots, p_k$ are affinely independent when no placement can be reconstructed from the others by a completed affine balance relation whose coefficients sum to the unit report. After coordinate projection, this is equivalent to linear independence of the displacement reports*

$$p_1 - p_0, \dots, p_k - p_0.$$

Definition 59.33 (k-simplex)**.** *A k-simplex is the completed cell*

$$[p_0, p_1, \dots, p_k]$$

formed from $k+1$ affinely independent placements together with all of its lower-dimensional faces and their incidence orientations.

Theorem 59.34 (Primitive Measure Theorem)**.** *Let a constructive k-primitive be required to satisfy all of the following:*

(i) it is finite and vertex-defined;

(ii) its boundary is closed;

(iii) it has positive k-dimensional measure;

(iv) it uses the minimum number of placements compatible with the preceding requirements.

Then the primitive is a k-simplex. Under maximal balance symmetry, it is a regular k-simplex.

Proof. At most k placements lie in an affine carrier of dimension at most $k-1$ and therefore cannot enclose positive k-measure. Hence at least $k+1$ affinely independent placements are required. Exactly $k+1$ affinely independent placements generate one k-simplex whose alternating face ledger closes and whose faithful coordinate determinant report is nonzero. Therefore the minimum finite closed positive-k-measure primitive is the k-simplex. Requiring every vertex and edge role to be symmetry-equivalent yields the regular simplex. □

Corollary 59.35 (Dimensional sequence)**.** *The minimum finite constructive closures are:*

dimension	*minimum closure*	*balanced closure*
0	*placement*	*placement*
1	*line segment*	*equal unit line*
2	*triangle*	*equilateral triangle*
3	*tetrahedron*	*regular tetrahedron*
k	*k-simplex*	*regular k-simplex*

59.8 Cell measure and overlap

Definition 59.36 (Primitive cell count)**.** *Fix a balanced positive k-primitive Δ_k as a unit cell. For a finite nonoverlapping complex K made from complete copies of Δ_k, its primitive cell count is*

$$N_k(K) := \#\{\textit{whole } \Delta_k \textit{ cells in } K\}.$$

Definition 59.37 (Constructive finite measure)**.** *If each primitive cell carries positive unit measure $\mu_k(\Delta_k)$ and interiors do not overlap, define*

$$\mu_k(K) = N_k(K)\,\mu_k(\Delta_k).$$

Shared lower-dimensional boundaries are incidence relations and are not counted as duplicate k-measure.

Theorem 59.38 (Finite additivity under interior disjointness)**.** *If finite complexes K_1 and K_2 have disjoint k-dimensional interiors, then*

$$\mu_k(K_1 \sqcup K_2) = \mu_k(K_1) + \mu_k(K_2).$$

Shared boundaries do not add positive k-measure.

Principle 59.39 (Overlap ledger requirement)**.** *When k-cells overlap in positive k-measure, source-preserving join alone overcounts geometric measure. A valid measure report must retain an overlap ledger or a disjoint decomposition. Extensional union, source multiplicity, and geometric measure are different projections.*

59.9 Boundary and measure preservation

Theorem 59.40 (Geometric ledger preservation)**.** *A report of a finite positive-measure complex is lossless for boundary and measure only when it retains:*

(i) whole cell identities and dimensions;

(ii) incidence and adjacency;

(iii) orientation;

(iv) exposed and internally cancelled boundary cells;

(v) overlap or disjointness information;

(vi) the primitive unit and normalization;

(vii) and the coordinate frame if a coordinate magnitude is reported.

Equal scalar measures do not imply equal boundaries, equal interiors, equal topology, or equal constructions.

60 Coordinate projection

Coordinates are now admissible because completed real reports, signed orientation, mappings, roots, and equality fibers have already been constructed. They are introduced as a reporting grammar over prior placement relations.

$$\boxed{\text{placement relation} \longrightarrow \text{reference carriers} \longrightarrow \text{coordinate assignment}}$$

60.1 Reference frames

Definition 60.1 (Coordinate reference frame)**.** *A d-coordinate reference frame is a report*

$$\mathcal{F} = (O; E_1, \ldots, E_d; U_1, \ldots, U_d),$$

where:

(i) O is a declared origin placement;

(ii) each E_j is an oriented reference carrier through O;

(iii) the E_j are independent under the declared geometric grammar;

(iv) each U_j is a positive unit relation on E_j.

The frame is itself a construction. Choosing an origin, orientation, number of axes, and units is not a neutral act of discovery; it is a declared reporting scaffold.

Definition 60.2 (Coordinate assignment)**.** *Let P be a family of placements resolvable relative to $\mathcal{F}$. A coordinate assignment is a map*

$$\kappa_{\mathcal{F}} : P \longrightarrow \mathbb{R}_{\mathrm{C}}^{d}$$

such that

$$\kappa_{\mathcal{F}}(p) = (x_1(p), \ldots, x_d(p))$$

reports the signed completed ratio of the displacement from O to p along each reference carrier.

The assignment reports a placement by d completed scalars. It does not replace the placement, its source, its admissible paths, or its incidence relations.

Definition 60.3 (Cartesian frame)**.** *A Cartesian frame is a coordinate frame whose reference carriers are pairwise orthogonal under a declared inner-product or right-angle report and whose units are independently repeatable.*

Orthogonality is therefore additional structure. Distinguishable placement alone does not generate perpendicular axes.

Principle 60.4 (Axes are not consumed space)**.** *An axis is a reference relation used to assign signed ratios. It need not be a material boundary, a primitive area carrier, or an independently consumed spatial direction. A coordinate plane is a report carrier constructed from two reference lanes; it is not the minimum positive-area primitive.*

60.2 Rectangular grids

Definition 60.5 (Finite rectangular grid report)**.** *Let $I_1, \ldots, I_d$ be finite indexed coordinate carriers. Their rectangular grid is the placement product*

$$G = I_1 \times \cdots \times I_d.$$

Every grid cell is addressed by one ordered d-tuple of index positions.

A grid is a cross-index report. It does not, by indexing alone, certify geometric adjacency, positive cell measure, or orthogonality. Those relations must be added.

Definition 60.6 (Geometric rectangular cell)**.** *A geometric rectangular cell is a grid cell together with declared orthogonal edge relations, positive unit lengths, boundary closure, and the appropriate positive-dimensional cell report.*

Thus a table of entries, an $m \times n$ matrix, and a geometric rectangle may share a two-index display while remaining different constructions.

Principle 60.7 (Data dimension and geometric dimension)**.** *The number of indices in a report is its data-index dimension. The number of independent positive-measure extension relations is its geometric dimension. A 3×3 matrix is ordinarily a two-index row–column ledger; it does not acquire a third geometric depth merely because each index has three values.*

60.3 Coordinate equality and its fibers

Definition 60.8 (Coordinate equality)**.** *For placements p, q in the domain of one frame, define*

$$p =_{\mathcal{F}} q \quad \Longleftrightarrow \quad \kappa_{\mathcal{F}}(p) = \kappa_{\mathcal{F}}(q).$$

Theorem 60.9 (Coordinate equality requires a faithfulness theorem)**.** *Coordinate equality implies placement identity only when $\kappa_{\mathcal{F}}$ is injective on the relevant placement domain. It implies full constructive identity only when every declared source, orientation, incidence, boundary, and history property is constant on the coordinate fibers.*

Proof. This is the projection-fiber preservation theorem of Section 54 applied to the coordinate map $\kappa_{\mathcal{F}}$. Equal coordinate tuples locate two objects in the same coordinate fiber. Any stronger property descends only if it is constant on that fiber. □

Example 60.10 (Same coordinate, different occurrence)**.** *Two source-distinct events may be reported at the same coordinate:*

$$\kappa_{\mathcal{F}}(e_1) = \kappa_{\mathcal{F}}(e_2), \qquad e_1 \neq_{\mathrm{C}} e_2.$$

The coordinate report records co-location, not occurrence identity.

60.4 Displacement and distance

Definition 60.11 (Coordinate displacement)**.** *For placements p, q, the coordinate displacement report is*

$$\Delta_{\mathcal{F}}(p, q) := \kappa_{\mathcal{F}}(q) - \kappa_{\mathcal{F}}(p) = (\Delta x_1, \ldots, \Delta x_d).$$

The displacement retains signed axis components but not necessarily the constructive path joining p to q.

Definition 60.12 (Cartesian distance)**.** *In a Cartesian frame, define the coordinate distance*

$$d_{\mathcal{F}}(p, q) := \sqrt{(\Delta x_1)^2 + \cdots + (\Delta x_d)^2},$$

using the completed nonnegative root constructed in Sections 51 and 57.

Theorem 60.13 (Distance is a projection)**.** *Equal coordinate distance does not imply equal displacement, path, orientation, boundary interaction, or source history.*

Proof. The distance map factors a displacement tuple through squaring, summation, and completed root. Sign and axis distribution are generally lost. Many distinct displacement tuples and arbitrarily many paths therefore occupy the same distance fiber. □

Principle 60.14 (Path length and endpoint distance)**.** *Endpoint distance is the magnitude of one coordinate displacement. Path length is the accumulated measure of a declared path ledger. They agree only for path families satisfying an additional minimizing or straightness theorem.*

60.5 Coordinate reports of area and volume

Let a triangle have one selected vertex p_0 and displacement reports

$$u = p_1 - p_0, \qquad v = p_2 - p_0.$$

In a two-coordinate frame write

$$u = (u_1, u_2), \qquad v = (v_1, v_2).$$

Definition 60.15 (Signed two-lane closure report)**.** *The signed coordinate closure of u and v is*

$$J_2(u, v) := u_1 v_2 - u_2 v_1.$$

The two terms report opposing row–column orientations. The closure is zero exactly when the two displacement reports are coordinate-dependent.

Theorem 60.16 (Triangle area as coordinate projection)**.** *For a faithfully embedded triangle,*

$$A_{\triangle} = \frac{1}{2} |J_2(u, v)|.$$

The parallelogram generated by u and v has coordinate area $|J_2(u, v)|$ and decomposes into two congruent triangular closures.

The formula measures one whole triangular primitive relative to a composite parallelogram carrier. It does not construct the triangle by halving an area occurrence.

For a tetrahedron with displacement reports u, v, w from one vertex, define the signed three-lane closure by the determinant report

$$J_3(u, v, w) := \det[u \; v \; w].$$

Theorem 60.17 (Tetrahedral volume as coordinate projection)**.** *For a faithfully embedded tetrahedron,*

$$V_{\text{tet}} = \frac{1}{6} |J_3(u, v, w)|.$$

The factor $1/6$ compares one tetrahedral closure with the composite parallelepiped carrier generated by the same three displacement lanes.

Theorem 60.18 (Simplex coordinate measure)**.** *For a k-simplex with displacement matrix M from one selected vertex,*

$$\mu_k(\Delta_k) = \frac{1}{k!} |\det M|.$$

This is a coordinate measure theorem over an already constructed simplex. It is not the primitive definition of simplex closure.

Principle 60.19 (Row–column closure reading)**.** *A determinant is a signed square row–column closure report. Rectangular data ledgers do not possess a whole determinant until a square closure cell or explicit closure repair has been selected. Principal minors report lower-order matched closure cells inside a larger square ledger.*

60.6 Bivector and orientation report

Definition 60.20 (Oriented area relation)**.** *Before taking magnitude, the ordered pair of displacement relations (u, v) determines an oriented area relation, denoted*

$$u \wedge v.$$

Reversing order reverses orientation:

$$v \wedge u = -(u \wedge v).$$

The bivector symbol records the oriented two-relation plane of construction. The scalar area is a later magnitude projection. Thus

$$\|u \wedge v\|$$

forgets orientation that $u \wedge v$ retains.

Principle 60.21 (A plane is not required before a bivector)**.** *Two independent directed relations construct an oriented two-lane closure relation. An infinite orthogonal coordinate plane is one possible reporting carrier for that relation, not its primitive prerequisite.*

60.7 Frame change and invariance

Definition 60.22 (Coordinate transition)**.** *For two frames $\mathcal{F}$ and $\mathcal{G}$ on a shared placement domain, a coordinate transition is a map*

$$T_{\mathcal{G}\leftarrow\mathcal{F}}$$

satisfying

$$\kappa_{\mathcal{G}} = T_{\mathcal{G}\leftarrow\mathcal{F}} \circ \kappa_{\mathcal{F}}.$$

Definition 60.23 (Frame-invariant property)**.** *A property is frame-invariant over a declared family of coordinate transitions when it is constant under every transition in that family.*

Theorem 60.24 (Coordinate invariance is a factorization claim)**.** *A geometric property can be inferred from coordinate reports independently of frame only after it is proved constant on the combined fibers of coordinate assignment and admissible frame change.*

This is another application of Section 54. Coordinate convenience is not an automatic preservation theorem.

60.8 What coordinate projection retains and discards

A coordinate tuple may retain:

(i) signed location relative to the selected origin and axes;

(ii) scalar ratios relative to selected units;

(iii) enough information to reconstruct coordinate displacement;

(iv) metric magnitude when a metric is declared;

(v) and algebraic closure values computed from the tuple.

Unless separately encoded, it may discard:

(i) occurrence identity and source;

(ii) construction path and order;

(iii) admissible versus forbidden routes;

(iv) primitive cell decomposition;

(v) orientation after magnitude projection;

(vi) boundary and adjacency;

(vii) overlap and multiplicity;

(viii) the reference-frame choice after normalization;

(ix) and the distinction between a primitive closure and a composite coordinate carrier.

Theorem 60.25 (Coordinate-report preservation)**.** *A coordinate representation is sufficient for a property family $\mathcal{P}$ exactly when every property in $\mathcal{P}$ is constant on every coordinate fiber. No stronger conclusion follows from the tuple alone.*

61 Orientation cycles, angle, circle, and π

The construction of angle and circle must begin with finite orientation transitions. Neither radians nor π may be inserted into the primitive definitions that are later used to derive them.

61.1 Orientation transitions

Definition 61.1 (Orientation state)**.** *An orientation state is a distinguishable directed relation relative to a declared construction. It need not initially be a real-number angle or a coordinate vector.*

Definition 61.2 (Hinge transition)**.** *A hinge transition is an ordered relation*

$$h : q \longrightarrow q'$$

that joins one orientation state to its successor while retaining the complete line relations incident to the hinge.

Definition 61.3 (Finite cyclic orientation ledger)**.** *For a positive integer $n \geq 3$, define the finite cyclic ledger*

$$O_n := \mathbb{Z}/n\mathbb{Z}.$$

Let $T_n = [1]$ be the positive successor transition. The states are

$$q_j := T_n^j, \qquad 0 \leq j < n.$$

The first positive return to the initial state occurs after exactly n transitions:

$$T_n^n = I, \qquad T_n^j \neq I \quad (0 < j < n).$$

This finite cycle uses integer quotient–remainder closure. It does not require a prior circle or real angle.

Definition 61.4 (Partial orientation cycle)**.** *A partial cycle is the retained state*

$$(k; n), \qquad 0 \le k < n,$$

meaning k completed successor transitions relative to one full n-transition return.

Definition 61.5 (Turn ratio)**.** *The normalized turn report of $(k; n)$ is the constructive ratio*

$$\mathrm{turn}(k; n) := \frac{k}{n}.$$

This ratio is a later scalar report of the finite cycle ledger.

Definition 61.6 (Directed turn)**.** *A reversed orientation cycle reports*

$$\mathrm{turn}_{-}(k; n) := -\frac{k}{n}.$$

The sign reports direction, not negative multiplicity.

61.2 Composition and winding

Theorem 61.7 (Finite angle composition)**.** *For partial cycles $(a; n)$ and $(b; n)$,*

$$a + b = qn + r, \qquad 0 \le r < n,$$

where q counts completed full turns and r is the retained terminal orientation state. Thus orientation composition is a quotient–remainder ledger before it is a scalar angle addition.

Definition 61.8 (Winding report)**.** *The quotient q in*

$$a = qn + r$$

is the winding report: the number of completed orientation cycles. The remainder r reports terminal orientation within the current cycle.

Two paths may have the same terminal orientation remainder and different winding quotients. Terminal angle equality therefore does not imply path or winding equality.

Definition 61.9 (Balanced constraint ring)**.** *A balanced n-constraint ring R_n is the cyclic incidence ledger with:*

(i) n complete equal-length line events;

(ii) orientation states $q_0, \ldots, q_{n-1}$ from O_n;

(iii) one equal successor hinge T_n between every adjacent pair;

(iv) and exact return after the nth transition.

Theorem 61.10 (Existence of balanced finite rings)**.** *For every integer $n \ge 3$, a balanced n-constraint ring exists as one exact cyclic incidence ledger. It is unique up to cyclic relabeling, reversal, and global orientation within the declared equal-transition grammar.*

Proof. Addition by [1] permutes the residue classes of $\mathbb{Z}/n\mathbb{Z}$ in one cycle of exact order n. Assign one complete equal line to each successor incidence. Any change of initial state is a cyclic relabeling; replacing [1] by $[-1]$ reverses orientation; a uniform global orientation change preserves incidence. □

A planar Euclidean realization of this ledger with equal side lengths and equal positive hinges is the ordinary regular n-gon. The native object is the finite equal-line, equal-transition cycle; the regular polygon is a coordinate report.

61.3 Native ring and circle classes

A finite balanced ring is closed, but it is not yet a circle. A circle report must describe what remains invariant as finite balanced rings become arbitrarily rich while no primitive line is divided below one completed contribution.

Definition 61.11 (Primitive triangle shell)**.** *Fix a positive unit line L and the balanced positive-area primitive Δ_L. A primitive triangle shell around a finite equal-line core is a finite nonoverlapping complex of complete copies of Δ_L joined by whole-edge adjacency, with exposed and shared edges retained.*

For every finite triangle shell,

$$3N = 2I + P,$$

where N is primitive-triangle count, I shared-edge count, and P exposed boundary count.

Definition 61.12 (Balanced ring lineage)**.** *A balanced ring lineage is an indexed family*

$$\mathcal{R} = (R_{n_j})_{j\in\mathbb{N}}$$

with strictly increasing completed constraint counts n_j, fixed primitive line type L, exact cyclic hinge ledgers, and retained shell data.

Definition 61.13 (Finite defect report)**.** *A finite defect report $e(R_n)$ is a nonnegative integer ledger recording declared departures from shell balance, such as bounded sixfold remainder, collision residue, source-capacity residue, or hinge dispersion.*

Definition 61.14 (Constructive circle lineage)**.** *A balanced ring lineage is a constructive circle lineage when, for every positive rational resolution ρ, there exists a finite stage J_ρ such that*

$$j \geq J_\rho \Longrightarrow \frac{e(R_{n_j})}{N(R_{n_j})} < \rho,$$

where $N(R_{n_j})$ is a positive completed shell-size report.

Definition 61.15 (Native constructive circle class)**.** *The native constructive circle class is the completion-equivalence class of constructive circle lineages under vanishing normalized defect and orientation-preserving cyclic correspondence.*

Principle 61.16 (No shrinking primitive)**.** *In the native circle lineage, the primitive line L remains a complete positive contribution. The transition is not $L \to 0$. Completed constraint and shell counts grow while every fixed finite defect becomes dominated by completed construction.*

Thus the native circle is a stable relational class of balanced finite rings. It is not inserted as a perfect curve before the finite constructions and is not identical to any one polygonal stage.

61.4 Coordinate normalization of ring lineages

To compare circumference ratios, a later coordinate grammar may normalize differently sized native rings to a common radial report. This does not retroactively divide the native primitive.

Definition 61.17 (Normalized Euclidean ring report)**.** *A normalized Euclidean report of a balanced ring is a coordinate realization whose vertices have completed distance 1 from a common center placement. Its reported circumdiameter is therefore 2.*

The side length in the normalized coordinate report may vary with n. That variation belongs to the coordinate scale map. The native ring may still have fixed primitive side type L.

Principle 61.18 (Normalization is not primitive refinement)**.** *Scaling a coordinate report changes the numerical unit assigned to the whole construction. It does not assert that a native line event has been physically or logically divided into fractional primitive occurrences.*

61.5 A noncircular polygon construction of π

We now construct the Euclidean circular closure constant without using a prior angle measure, sine function, radian, or value of π.

Begin with a normalized regular square whose four vertices have coordinate distance 1 from its center. Its side report is

$$s_4 = \sqrt{2}.$$

Given the side report s_n of a normalized regular n-ring, define its radial midpoint report

$$a_n := \sqrt{1 - \frac{s_n^2}{4}}.$$

The balanced $2n$-ring is obtained by inserting one normalized midpoint-direction vertex between every adjacent pair. Its side report is

$$s_{2n} := \sqrt{2 - 2a_n} = \sqrt{2 - 2\sqrt{1 - \frac{s_n^2}{4}}}.$$

All roots are completed nonnegative roots already constructed in Part XIII.

Definition 61.19 (Inner perimeter report)**.** *For $n = 2^k$ with $k \geq 2$, define*

$$P_n := ns_n.$$

This is the perimeter of the normalized vertex ring.

Definition 61.20 (Outer tangent perimeter report)**.** *Define the tangent-side report*

$$t_n := \frac{s_n}{a_n}$$

and the outer perimeter

$$Q_n := nt_n = \frac{ns_n}{a_n}.$$

The outer line at each side is perpendicular to the corresponding radial midpoint relation and touches the unit radial carrier at that midpoint.

The terms *inner* and *outer* refer to finite coordinate comparison ledgers around the common center and radial unit. No completed circle is assumed in their construction.

Lemma 61.21 (Side contraction)**.** *For every doubling stage,*

$$0 < s_{2n}^2 = 2\left(1 - \sqrt{1 - \frac{s_n^2}{4}}\right) \leq \frac{s_n^2}{2}.$$

Consequently $s_{2^k} \to 0$ as a normalized coordinate report.

Proof. For $0 \le x \le 1$, $1 - \sqrt{1-x} \le x$. Substitute $x = s_n^2/4$ and multiply by 2. Iteration gives geometric domination of $s_{2^k}^2$. □

This is a coordinate-scale convergence statement. It does not say that the fixed native primitive line tends to literal zero.

Lemma 61.22 (Nested perimeter bounds)**.** *The finite algebraic constructions above satisfy*

$$P_n < P_{2n} < Q_{2n} < Q_n$$

for every doubling stage $n \ge 4$.

Proof. Each inserted midpoint-direction vertex replaces one chord by two shorter chords whose joined length is strictly greater than the original chord, giving $P_n < P_{2n}$. The corresponding tangent construction replaces each outer side by two sides whose total length is smaller while remaining outside the vertex ring, giving $Q_{2n} < Q_n$. In coordinates, both inequalities reduce, after division by positive factors and squaring positive terms, to $0 < s_n < 2$. The strict inner–outer inequality follows from $0 < a_n < 1$, hence $s_n < s_n/a_n$. □

Lemma 61.23 (Vanishing perimeter gap)**.**

$$Q_n - P_n = P_n \left(\frac{1}{a_n} - 1 \right) \longrightarrow 0$$

along the doubling lineage.

Proof. The outer perimeters are decreasing and positive, so $P_n < Q_n \le Q_4$ gives a uniform finite bound on P_n. Side contraction gives $s_n \to 0$, hence

$$a_n = \sqrt{1 - s_n^2/4} \to 1.$$

Therefore $a_n^{-1} - 1 \to 0$, and the product with bounded P_n tends to zero. □

Definition 61.24 (Constructive Euclidean pi)**.** *The nested real intervals*

$$I_n := \left[\frac{P_n}{2}, \frac{Q_n}{2} \right]$$

have vanishing width. By nested-interval completeness, they contain exactly one completed real. Define

$$\boxed{\pi_{\mathrm{C}} := \bigcap_{k \ge 2} \left[\frac{P_{2^k}}{2}, \frac{Q_{2^k}}{2} \right].}$$

The denominator 2 is the normalized circumdiameter. Hence π_{C} is the completed perimeter-to-diameter closure constant selected by finite balanced polygon bounds.

Theorem 61.25 (Existence and uniqueness of the circular closure constant)**.** *The report π_{C} exists uniquely in $\mathbb{R}_{\mathrm{C}}$. It is independent of any finite doubling stage and is unchanged by uniform coordinate scaling.*

Proof. The intervals I_n are nested, nonempty, and have widths $(Q_n - P_n)/2 \to 0$. Section 58 therefore supplies one completed real in their intersection. If every coordinate length is scaled by a positive factor λ, both perimeter and diameter are multiplied by λ, so their ratio and every interval I_n remain unchanged. □

Principle 61.26 (Native circle versus coordinate pi)**.** *The native circle class is a stable lineage of finite incidence and shell constructions. The scalar π_{C} is a completed Euclidean coordinate invariant of normalized ring reports. The latter is a projection of the former together with a coordinate metric and normalization; it is not the primitive constructor of the circle class.*

Within the Euclidean coordinate grammar, this nested regular-polygon construction is the conventional circumference-to-diameter constant customarily denoted π. The subscript C records its constructive derivation in the present manuscript.

61.6 Circle report

Definition 61.27 (Completed coordinate circle report)**.** *A completed Euclidean circle report of radius $r > 0$ is the completion class of normalized balanced ring reports scaled by r, with vanishing inner–outer radial and boundary gap. Its circumference report is*

$$C(r) := 2\pi_{\mathrm{C}} r.$$

Its diameter report is

$$D(r) := 2r.$$

Thus

$$\frac{C(r)}{D(r)} = \pi_{\mathrm{C}}.$$

This definition is not circular: the ring lineage and the nested perimeter bounds construct π_{C} before the scalar circumference formula is used.

Theorem 61.28 (Scale invariance of circular ratio)**.** *For every positive completed radius r,*

$$\frac{C(r)}{D(r)} = \pi_{\mathrm{C}}.$$

The ratio does not depend on r.

61.7 Angle as a completed report

Definition 61.29 (Turn angle)**.** *The primitive angle report associated with k transitions of an n-cycle is the directed turn relation*

$$\alpha_{k,n} := (k; n),$$

with normalized turn value k/n.

Definition 61.30 (Radian projection)**.** *After π_{C} has been constructed, the radian projection of the directed turn relation is*

$$\theta_{k,n} := 2\pi_{\mathrm{C}} \frac{k}{n}.$$

One full positive turn reports $2\pi_{\mathrm{C}}$, a reversal reports π_{C}, and one quarter turn reports $\pi_{\mathrm{C}}/2$.

Radian measure is therefore a circumference-normalized coordinate report of an already constructed orientation ratio. It is not the primitive source of orientation.

Theorem 61.31 (Angle composition with winding)**.** *Let directed turn reports have integer transition counts a and b relative to an n-cycle. If*

$$a + b = qn + r, \qquad 0 \le r < n,$$

then their radian projections satisfy

$$\theta_a + \theta_b = 2\pi_{\mathrm{C}} q + \theta_r.$$

The quotient q retains full winding, while θ_r is the terminal angle modulo one turn.

61.8 Chord, arc, and boundary distinctions

Definition 61.32 (Chord report)**.** *A chord is a direct line relation between two placements on one completed circle report.*

Definition 61.33 (Arc report)**.** *An arc is an oriented boundary sublineage between two circle placements, retaining orientation and turn ratio.*

A chord and an arc may share endpoints while remaining constructively different. Endpoint coordinates do not identify path type.

Definition 61.34 (Arc-length projection)**.** *For a circle report of radius r and directed turn ratio τ, define the arc-length projection*

$$L_{\mathrm{arc}} = 2\pi_{\mathrm{C}} r\,|\tau|.$$

The sign of τ remains available as orientation even when length takes magnitude.

61.9 Orientation and circle preservation theorem

Theorem 61.35 (Orientation-cycle preservation)**.** *A scalar angle report is lossless for a finite orientation construction only when it retains or permits recovery of:*

(i) the full-cycle carrier n or its completed circle normalization;

(ii) direction;

(iii) terminal remainder;

(iv) winding quotient;

(v) hinge order;

(vi) and the equivalence relation under which cyclic relabeling is ignored.

Equal angles modulo $2\pi_{\mathrm{C}}$ do not imply equal paths or equal winding histories.

Theorem 61.36 (Circle-report preservation)**.** *A completed scalar radius and center determine a Euclidean coordinate circle fiber. They do not, by themselves, recover the native primitive line type, finite ring lineage, shell decomposition, orientation history, or approximation rule. Any property depending on those data descends to the scalar circle report only after a projection-fiber constancy proof.*

61.10 Part XIV summary

The geometric construction of Sections 59–61 is

$$\begin{aligned}\text{placements} &\longrightarrow \text{oriented incidence} \longrightarrow \text{boundary closure}\\ &\longrightarrow \text{interior/exterior separation} \longrightarrow \text{positive simplex measure}\\ &\longrightarrow \text{coordinate projection} \longrightarrow \text{metric and determinant reports}\\ &\longrightarrow \text{orientation cycles} \longrightarrow \text{balanced ring lineages}\\ &\longrightarrow \text{circle completion and } \pi_{\mathrm{C}}.\end{aligned}$$

The essential conclusions are:

(1) open incidence retains unmatched boundary obligations;

(2) closed incidence produces a typed boundary zero rather than literal absence;

(3) boundary-of-boundary closure follows from whole-face cancellation with opposite orientation;

(4) interior and exterior are separation relations produced by boundary closure;

(5) positive k-measure requires a completed k-dimensional contribution or a declared completion supplying one;

(6) the minimum finite vertex-defined positive-k-measure primitive is the k-simplex;

(7) the triangle and tetrahedron are whole minimum closures, while squares and cubes are composite coordinate carriers;

(8) coordinates are later assignments relative to origins, axes, units, and metrics;

(9) coordinate equality, distance, area, and volume are projections whose fibers require preservation audits;

(10) a bivector retains oriented area relation before scalar magnitude;

(11) a finite angle is first a partial orientation-cycle ledger and only later a real or radian report;

(12) a native circle is a stable class of balanced finite ring lineages, not one polygon and not a shrinking primitive;

(13) normalized inner and outer polygon ledgers construct a unique completed circumference-to-diameter constant π_{C} without presupposing radians or a prior value of π;

(14) and equal scalar circle reports do not recover construction lineage unless the relevant properties are constant on their fibers.

The next part should compile the previously constructed arithmetic and geometric operations into algebraic carriers: identities, inverses, closure systems, groups, rings, fields, and complex oriented-pair reports.

Part XV
Algebraic Carriers

62 Identity, inverse, and algebraic closure

The arithmetic, mapping, equality, completion, and geometric constructions already established can now be compiled into algebraic carriers. The order of dependence matters. A group, ring, or field is not introduced as a primitive universe containing unexplained operations. It is a report system whose carrier, operations, identities, inverses, closure rules, and equality relation have each already been constructed or explicitly declared.

62.1 Algebraic carriers are typed report systems

Definition 62.1 (Algebraic carrier specification)**.** *An* algebraic carrier specification *is a report*

$$\mathcal{A} = (A, \mathcal{O}, \mathcal{E}, \mathcal{D}, \Gamma),$$

where:

(i) A *is a declared carrier of report objects;*

(ii) $\mathcal{O}$ *is a family of total or partial operations on those reports;*

(iii) $\mathcal{E}$ *is the equality or congruence grammar under which outputs are compared;*

(iv) $\mathcal{D}$ *records the admissible domains of partial operations;*

(v) Γ *records any source, orientation, quotient, boundary, completion, or construction data retained by the system.*

The notation

$$(A, +, \cdot)$$

is therefore a compressed report. It is meaningful only after the meanings of A, $+$, $\cdot$, and equality on A have been fixed.

Principle 62.2 (No primitive algebraic universe)**.** *An algebraic structure may summarize previously constructed relations, but it may not be used retroactively to define the objects or operations from which it was compiled.*

Remark 62.3 (Relation to the earlier PCA manuscript)**.** *The earlier* Primitive Constructive Algebra *manuscript introduced construction ledgers, boundary lanes, graded equality, and the projection-collapse principle. In the present manuscript those ideas are no longer primitive assumptions. They are recompiled from the prior PCL constructions of reports, occurrence identity, signed orientation, quotient–remainder states, mappings, equality fibers, completion, boundary, and measure. A PCA-style ledger is consequently one admissible choice of* Γ*, not the foundation of the carrier itself.*

62.2 Closure as an operation obligation

Definition 62.4 (Operational closure)**.** *Let $\star$ be a k-ary operation with declared admissible domain $D_\star \subseteq A^k$. The carrier A is* closed under $\star$ *when*

$$(a_1, \ldots, a_k) \in D_\star \quad \Longrightarrow \quad \star(a_1, \ldots, a_k) \in A.$$

If $D_\star = A^k$, the operation is total on A. Otherwise it is partial.

Closure is not a claim that every attempted construction succeeds. It states that every *admissible* operation returns a report in the declared carrier.

Definition 62.5 (Typed failure and typed residual)**.** *For a partial constructive operation, an inadmissible input may return a typed failure marker $\bot_\star$, while an incomplete but meaningful construction may return a typed residual report $R_\star$. Neither is silently identified with the carrier identity unless a theorem proves that identification.*

Principle 62.6 (Closure is relative)**.** *The same operation may be closed on one carrier and not on another. For example, natural-number subtraction is not closed on $\mathbb{N}_{\mathrm{C}}$, signed subtraction is closed on $\mathbb{Z}_{\mathrm{C}}$, and reciprocal formation is closed only on the declared nonzero or apart subcarrier.*

62.3 Identity reports

Definition 62.7 (Left and right identity)**.** *Let $\star$ be a binary operation on A. A report $e_L \in A$ is a* left identity *when*

$$e_L \star a =_{\mathcal{E}} a$$

for every admissible $a \in A$. A report $e_R \in A$ is a right identity *when*

$$a \star e_R =_{\mathcal{E}} a$$

for every admissible $a \in A$. A two-sided identity satisfies both conditions.

The equality symbol is part of the definition. An object may be an identity under scalar equality while retaining a nontrivial source ledger under constructive equality.

Theorem 62.8 (Uniqueness of a two-sided identity)**.** *If e and e' are two-sided identities for the same operation and equality grammar, then*

$$e =_{\mathcal{E}} e'.$$

Proof. Using e as a left identity and e' as a right identity,

$$e \star e' =_{\mathcal{E}} e' \qquad \text{and} \qquad e \star e' =_{\mathcal{E}} e.$$

Transitivity gives $e =_{\mathcal{E}} e'$. □

Definition 62.9 (Additive and multiplicative identities)**.** *The additive identity is the typed zero report 0_A satisfying*

$$0_A + a =_A a = a + 0_A.$$

The multiplicative identity is the typed unit report 1_A satisfying

$$1_A a =_A a = a 1_A.$$

The zero and unit are not imported from an external number system. In each carrier they are the identities compiled from the earlier no-contribution closure and one-copy constructions.

62.4 Inverse reports

Definition 62.10 (Left and right inverse)**.** *Let e be the identity for $\star$. A report b is a left inverse of a when*

$$b \star a =_{\mathcal{E}} e,$$

and a right inverse when

$$a \star b =_{\mathcal{E}} e.$$

A two-sided inverse satisfies both.

Theorem 62.11 (Uniqueness of a two-sided inverse)**.** *In an associative carrier with identity e, a two-sided inverse of a is unique under the declared equality.*

Proof. If b and c are both two-sided inverses of a, then

$$b =_{\mathcal{E}} b \star e =_{\mathcal{E}} b \star (a \star c) =_{\mathcal{E}} (b \star a) \star c =_{\mathcal{E}} e \star c =_{\mathcal{E}} c.$$

□

Definition 62.12 (Additive inverse)**.** *The additive inverse $-a$ is the orientation-reversed report already constructed in the signed carrier, satisfying*

$$a + (-a) =_A 0_A.$$

Definition 62.13 (Multiplicative inverse)**.** *A multiplicative inverse a^{-1} is a report satisfying*

$$aa^{-1} =_A 1_A = a^{-1}a.$$

Its construction is admissible only on the subcarrier for which inverse existence has been proved.

Principle 62.14 (Inverse existence is not symbolic permission)**.** *Writing a^{-1} does not construct an inverse. It references a previously established inverse report and its admissibility conditions.*

62.5 Semigroups, monoids, and groups

Definition 62.15 (Semigroup)**.** *A* constructive semigroup *is a carrier with one closed associative binary operation.*

Definition 62.16 (Monoid)**.** *A* constructive monoid *is a constructive semigroup with a two-sided identity.*

Definition 62.17 (Group)**.** *A* constructive group *is a constructive monoid in which every carrier report has a two-sided inverse.*

Definition 62.18 (Abelian group)**.** *A constructive group is* abelian *when*

$$a \star b =_A b \star a$$

for every $a, b \in A$.

The equality here may be scalar or quotient equality even when the transposed source ledgers differ. Commutativity is therefore a theorem about the selected carrier report, not a claim that the two evaluation histories are identical.

Theorem 62.19 (Previously constructed additive carriers)**.** *Under their declared report equalities:*

(i) $(\mathbb{N}_{\mathrm{C}}, +, 0)$ *is a commutative monoid;*

(ii) $(\mathbb{Z}_{\mathrm{C}}, +, 0, -)$ *is an abelian group;*

(iii) $(\mathbb{Q}_{\mathrm{C}}, +, 0, -)$ *is an abelian group;*

(iv) $(\mathbb{R}_{\mathrm{C}}, +, 0, -)$ *is an abelian group.*

Proof. Closure and associativity of source-preserving addition were established before signed orientation. The typed zero is the no-contribution identity. Signed orientation supplies additive inverses on $\mathbb{Z}_{\mathrm{C}}$, and ratio and Cauchy completion arithmetic extend those operations to $\mathbb{Q}_{\mathrm{C}}$ and $\mathbb{R}_{\mathrm{C}}$. The operation-preservation theorems for ratio equality and Cauchy equivalence make the operations well defined on the corresponding quotient carriers. □

62.6 Multiplicative carriers and units

Definition 62.20 (Unit of a multiplicative carrier)**.** *A report $u \in A$ is a* unit *when it has a multiplicative inverse in A. The unit carrier is*

$$A^{\times} := \{u \in A : u^{-1} \text{ exists in } A\}.$$

Theorem 62.21 (Units form a group)**.** *If multiplication on A is associative with identity 1_A, then $(A^{\times}, \cdot, 1_A)$ is a group.*

Proof. The identity is a unit. If u and v are units, then

$$(uv)^{-1} = v^{-1}u^{-1},$$

so the product is a unit. Associativity is inherited from A, and inverse uniqueness was proved above. □

For commutative multiplication, the unit group is abelian.

62.7 Semirings, rings, and integral domains

Definition 62.22 (Constructive semiring)**.** *A* constructive semiring *is a carrier A with operations $+$ and $\cdot$ such that:*

(i) $(A, +, 0_A)$ *is a commutative monoid;*

(ii) $(A, \cdot, 1_A)$ *is a monoid;*

(iii) multiplication distributes over addition;

(iv) $0_A a = a 0_A = 0_A$.

Definition 62.23 (Constructive ring)**.** *A* constructive ring *is a constructive semiring whose additive monoid is an abelian group.*

Definition 62.24 (Commutative ring)**.** *A constructive ring is commutative when multiplication is commutative under the carrier equality.*

Definition 62.25 (Zero divisor)**.** *A nonzero report a is a* zero divisor *when there exists a nonzero report b such that*

$$ab = 0_A.$$

Definition 62.26 (Constructive integral domain)**.** *A* constructive integral domain *is a commutative ring with $1_A \neq 0_A$ and no nonzero zero divisors.*

Theorem 62.27 (Compiled arithmetic structures)**.** *Under finite count, signed, ratio, and completed-real equality:*

(i) $\mathbb{N}_{\mathrm{C}}$ *is a commutative semiring;*

(ii) $\mathbb{Z}_{\mathrm{C}}$ *is a commutative ring and an integral domain;*

(iii) $\mathbb{Q}_{\mathrm{C}}$ *is a commutative ring and an integral domain;*

(iv) $\mathbb{R}_{\mathrm{C}}$ *is a commutative ordered ring and an integral domain under the completed-real equality and order already constructed.*

The theorem compiles earlier operation laws. It does not replace their construction ledgers.

62.8 Fields and apartness

Mere negation of equality is not always enough to execute a reciprocal construction in a constructive carrier. The inverse algorithm requires positive separation from zero.

Definition 62.28 (Apartness from zero)**.** *For an ordered completed carrier, write*

$$a \# 0_A$$

when there exists a positive rational resolution ρ such that

$$|a| \geq \rho.$$

This is a positive witness that a is separated from zero.

Definition 62.29 (Constructive field)**.** *A* constructive field *is a commutative ring with $1 \neq 0$ together with an apartness relation $\#$ such that every $a\#0$ has a multiplicative inverse.*

Theorem 62.30 (Rational field)**.** *Every nonzero finite constructive rational is apart from zero and has a finite reciprocal. Hence $\mathbb{Q}_{\mathrm{C}}$ is a constructive field.*

Proof. A nonzero rational has a reduced finite numerator with positive whole magnitude and a positive denominator. Its magnitude is bounded below by the reciprocal of that denominator. The reciprocal construction of Sections 36–38 therefore applies. □

Theorem 62.31 (Completed-real field under apartness)**.** *Every completed real report $x\#0$ has a completed reciprocal. Thus $\mathbb{R}_{\mathrm{C}}$ is a constructive field relative to the apartness relation.*

Proof. Section 58 constructed the reciprocal of every completed report positively separated from zero. Representative independence follows from the Cauchy-fiber preservation theorem. □

Remark 62.32 (Inequality and invertibility)**.** *In a classical logic one often replaces $x\#0$ by $x \neq 0$. PCL retains the distinction. A negative statement that equality has not been proved is not automatically the positive finite information needed to build an inverse.*

62.9 Congruences and quotient carriers

Definition 62.33 (Algebraic congruence)**.** *An equivalence relation $\sim$ on an algebraic carrier is a* congruence *for a k-ary operation $\star$ when*

$$a_j \sim b_j \text{ for every } j \quad \Longrightarrow \quad \star(a_1, \ldots, a_k) \sim \star(b_1, \ldots, b_k).$$

Theorem 62.34 (Operation descent to a quotient)**.** *An operation descends to a well-defined operation on the quotient carrier $A/\sim$ exactly when $\sim$ is a congruence for that operation.*

Proof. A quotient operation must be independent of the chosen representative. This is exactly the stated congruence condition. It is the operation form of the projection-fiber factorization theorem from Section 54. □

Example 62.35 (Residue ring as a quotient report)**.** *For a positive modulus n, integer congruence modulo n is a congruence for addition and multiplication. Therefore the residue carrier*

$$\mathbb{Z}_{\mathrm{C}}/n\mathbb{Z}_{\mathrm{C}}$$

inherits ring operations. The quotient discards the integer quotient lane unless that lane is retained in a larger boundary ledger.

Principle 62.36 (Quotient carrier discipline)**.** *A quotient carrier is valid for exactly those operations and predicates proved constant on its fibers. It does not make the erased source distinctions unreal.*

62.10 Homomorphisms and isomorphisms

Definition 62.37 (Homomorphism)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be carriers with corresponding operations. A mapping*

$$F : A \to B$$

is a homomorphism *when it preserves every declared operation and identity, for example*

$$F(a + a') = F(a) + F(a'), \qquad F(aa') = F(a)F(a'), \qquad F(0_A) = 0_B, \qquad F(1_A) = 1_B.$$

Definition 62.38 (Isomorphism)**.** *A bijective homomorphism whose inverse is also a homomorphism is an* isomorphism*. It certifies structural equivalence under the declared algebraic grammar, not necessarily source-ledger identity.*

Definition 62.39 (Kernel and image)**.** *For an additive homomorphism $F : A \to B$, define*

$$\ker F := \{a \in A : F(a) = 0_B\}, \qquad \operatorname{im} F := \{F(a) : a \in A\}.$$

The kernel is a zero-output fiber in the input carrier. It is not the same object as a complete evaluation ledger zero, which will be constructed in Section 64.

62.11 Operational closure versus algebraic closure

The phrase *algebraic closure* is used in two different senses and must be separated.

Definition 62.40 (Operationally closed carrier)**.** *A carrier is operationally closed under an operation when every admissible use of that operation remains in the carrier.*

Definition 62.41 (Polynomial report)**.** *A finite polynomial report over a commutative ring A is a finite coefficient ledger*

$$p(T) = a_0 + a_1T + \cdots + a_nT^n$$

with each $a_j \in A$ and finite operational depth.

Definition 62.42 (Algebraic element)**.** *Let B extend A. A report $\alpha \in B$ is* algebraic over A *when some nonzero finite polynomial report $p \in A[T]$ satisfies*

$$p(\alpha) = 0_B.$$

Definition 62.43 (Algebraically closed field)**.** *A field carrier F is* algebraically closed *when every nonconstant finite polynomial report in $F[T]$ has at least one root in F.*

This is a later completeness property of a field with respect to polynomial root obligations. It is not the same as closure under addition or multiplication.

Definition 62.44 (Adjoining a root)**.** *If $p(T) \in A[T]$ has no root in A, an extension may be constructed by adjoining a formal report α and imposing the smallest operation congruence containing*

$$p(\alpha) \sim 0.$$

The resulting quotient is admissible only after the congruence and noncollapse conditions have been proved.

The next section applies this compiler to the polynomial

$$T^2 + 1,$$

not by importing a mysterious scalar square root of -1, but by constructing an oriented two-lane report whose transverse unit squares to orientation reversal.

62.12 Section 62 preservation theorem

Theorem 62.45 (Algebraic-carrier preservation)**.** *A compressed algebraic carrier report is sufficient for a property P only when:*

(i) every operation used by P is closed on its declared admissible domain;

(ii) the carrier equality is a congruence for those operations;

(iii) identities and inverses are interpreted relative to that equality;

(iv) quotient projections preserve P on every fiber;

(v) and every partial operation retains its admissibility witness.

Without these conditions, equal algebraic outputs need not certify equal constructions.

63 Complex and oriented-pair reports

The completed real carrier supplies signed longitudinal magnitudes. Complex construction requires one additional distinguishable orientation lane transverse to the first. This transverse lane is not a second time, a hidden physical dimension, or a Cartesian axis inserted into reality. It is a retained algebraic role whose interaction law records quarter-turn orientation.

63.1 The oriented two-lane carrier

Definition 63.1 (Longitudinal and transverse basis roles)**.** *Let* $\mathbf{e}$ *denote the longitudinal unit role and let* $\mathbf{j}$ *denote a transverse unit role. They satisfy the declared orientation relations*

$$\mathbf{ee} = \mathbf{e}, \qquad \mathbf{ej} = \mathbf{je} = \mathbf{j}, \qquad \mathbf{j}^2 = -\mathbf{e}.$$

The last relation states that two successive positive quarter-turn orientation operations produce longitudinal reversal.

Definition 63.2 (Oriented real pair)**.** *An* oriented real pair *is a report*

$$z = a\mathbf{e} + b\mathbf{j}, \qquad a, b \in \mathbb{R}_{\mathrm{C}}.$$

Its pair notation is

$$z = (a, b)_{\mathbf{e},\mathbf{j}}.$$

The first lane is longitudinal and the second is transverse.

Definition 63.3 (Constructive complex carrier)**.** *The* constructive complex carrier *is*

$$\mathbb{C}_{\mathrm{C}} := \mathbb{R}_{\mathrm{C}}\mathbf{e} \oplus \mathbb{R}_{\mathrm{C}}\mathbf{j},$$

together with the pair operations derived below and completed-real equality in each lane.

The direct-sum symbol reports two retained roles. It does not mean that equal values in the two lanes become the same occurrence.

63.2 The complex unit as orientation operator

Definition 63.4 (Complex orientation unit)**.** *Define*

$$\mathbf{i} := 0\mathbf{e} + 1\mathbf{j} = (0, 1).$$

Then

$$\mathbf{i}^2 = -1,$$

where $1 = (1, 0)$ *and* $-1 = (-1, 0)$ *are the embedded real unit reports.*

Theorem 63.5 (Quarter-turn action)**.** *Multiplication by* $\mathbf{i}$ *acts on oriented pairs by*

$$\mathbf{i}(a, b) = (-b, a).$$

Consequently,

$$\mathbf{i}^2(a, b) = (-a, -b), \qquad \mathbf{i}^3(a, b) = (b, -a), \qquad \mathbf{i}^4(a, b) = (a, b).$$

Proof. The multiplication law below gives

$$(0, 1)(a, b) = (-b, a).$$

Repeating the operation yields the remaining orientation states. □

Thus $\mathbf{i}$ is not introduced as an unexplained number whose square happens to be negative. It is the algebraic report of a transverse quarter-turn generator whose second application reverses the longitudinal orientation.

63.3 Complex addition

Definition 63.6 (Oriented-pair addition)**.** *For*

$$z = (a, b), \qquad w = (c, d),$$

define

$$z + w := (a + c, b + d).$$

Definition 63.7 (Complex additive zero and inverse)**.** *Define*

$$0_{\mathbb{C}} := (0_{\mathbb{R}}, 0_{\mathbb{R}})$$

and

$$-z := (-a, -b).$$

Then

$$z + 0_{\mathbb{C}} = z, \qquad z + (-z) = 0_{\mathbb{C}}.$$

Theorem 63.8 (Complex additive group)**.** $(\mathbb{C}_{\mathrm{C}}, +, 0_{\mathbb{C}})$ *is an abelian group under completed pair equality.*

Proof. Every group law follows componentwise from the completed-real additive group. □

63.4 Complex multiplication

Distribute the oriented pair product while retaining the relation $\mathbf{j}^2 = -\mathbf{e}$:

$$(a\mathbf{e} + b\mathbf{j})(c\mathbf{e} + d\mathbf{j}) = ac\mathbf{e} + ad\mathbf{j} + bc\mathbf{j} + bd\mathbf{j}^2.$$

Therefore

$$(a\mathbf{e} + b\mathbf{j})(c\mathbf{e} + d\mathbf{j}) = (ac - bd)\mathbf{e} + (ad + bc)\mathbf{j}.$$

Definition 63.9 (Complex multiplication)**.** *Define*

$$(a, b)(c, d) := (ac - bd, ad + bc).$$

Theorem 63.10 (Complex ring laws)**.** *Complex multiplication is associative, commutative, and distributive over complex addition. Its identity is*

$$1_{\mathbb{C}} := (1, 0).$$

Hence $\mathbb{C}_{\mathrm{C}}$ *is a commutative ring.*

Proof. The multiplication law is the finite distributive expansion of products in the real commutative ring with every occurrence of $\mathbf{j}^2$ reduced to $-\mathbf{e}$. Reassociation, transposition, and distribution are therefore inherited from the real carrier and the declared basis relation. The identity calculation is immediate:

$$(a, b)(1, 0) = (a, b).$$

□

63.5 Quadratic-extension report

The oriented-pair construction has an equivalent polynomial form.

Theorem 63.11 (Pair carrier as a quadratic quotient)**.** *Let $\mathbb{R}_{\mathrm{C}}[T]$ be the finite polynomial carrier over the completed reals. The congruence generated by*

$$T^2 + 1 \sim 0$$

reduces every polynomial report to one representative of the form

$$a + bT.$$

The mapping

$$a + bT \longmapsto (a, b)$$

is a ring isomorphism from

$$\mathbb{R}_{\mathrm{C}}[T]/(T^2 + 1)$$

to $\mathbb{C}_{\mathrm{C}}$.

Proof. Repeated replacement of T^2 by -1 reduces every higher power to a real or transverse term. Addition and multiplication agree with the pair formulas. The quotient congruence makes the result independent of the chosen polynomial representative. □

This theorem does not make the polynomial quotient more primitive than the oriented pair. It proves that two compiled reports preserve the same ring structure.

63.6 Real embedding and lane projections

Definition 63.12 (Real embedding)**.** *Define*

$$\eta : \mathbb{R}_{\mathrm{C}} \to \mathbb{C}_{\mathrm{C}}, \qquad \eta(a) := (a, 0).$$

Theorem 63.13 (Real embedding preservation)**.** *The mapping η is an injective ring homomorphism.*

Definition 63.14 (Longitudinal and transverse projections)**.** *For $z = (a, b)$, define*

$$\mathrm{Re}_{\mathrm{C}}(z) := a, \qquad \mathrm{Im}_{\mathrm{C}}(z) := b.$$

These projections report the two oriented lanes. They do not recover the source construction by which the pair was produced.

63.7 Conjugation and reflection

Definition 63.15 (Complex conjugation)**.** *For $z = (a, b)$, define*

$$\overline{z} := (a, -b).$$

Conjugation preserves the longitudinal lane and reverses the transverse orientation. It is therefore the algebraic reflection across the retained longitudinal role.

Theorem 63.16 (Conjugation laws)**.** *For all $z, w \in \mathbb{C}_{\mathrm{C}}$,*

$$\overline{z+w} = \overline{z} + \overline{w},$$

$$\overline{zw} = \overline{z}\,\overline{w},$$

$$\overline{\overline{z}} = z,$$

and

$$\overline{\mathbf{i}} = -\mathbf{i}.$$

Proof. Each identity follows from the pair formulas and signed real arithmetic. □

Definition 63.17 (General line reflection)**.** *For a unit complex orientation report u satisfying $u\overline{u} = 1$, define reflection across the longitudinal line rotated by u as*

$$\mathrm{Ref}_u(z) := u\,\overline{u^{-1}z}.$$

This definition compiles reflection from rotation, conjugation, and inverse. The simpler conjugation map is the case $u = 1$.

63.8 Norm and magnitude

Definition 63.18 (Squared norm)**.** *For $z = (a, b)$, define*

$$N(z) := z\overline{z} = a^2 + b^2.$$

The result is embedded in the nonnegative completed-real lane.

Theorem 63.19 (Norm properties)**.** *For all $z, w \in \mathbb{C}_{\mathrm{C}}$:*

(i) $N(z) \geq 0$*;*

(ii) $N(z) = 0$ *if and only if* $z = 0_{\mathbb{C}}$*;*

(iii) $N(zw) = N(z)N(w)$*;*

(iv) $N(\overline{z}) = N(z)$*.*

Proof. Nonnegativity follows from real squares. If $a^2 + b^2 = 0$, nonnegativity forces $a = b = 0$ in the completed-real integral domain. Multiplicativity follows from

$$N(zw) = zw\overline{zw} = zw\overline{z}\,\overline{w} = z\overline{z}\,w\overline{w}.$$

Conjugation invariance is immediate. □

Definition 63.20 (Complex magnitude)**.** *The* magnitude *of z is the nonnegative completed root*

$$|z| := \sqrt{N(z)} = \sqrt{a^2 + b^2}.$$

Magnitude is a later real projection of the pair. It forgets orientation and phase.

Theorem 63.21 (Magnitude laws)**.** *For all $z, w \in \mathbb{C}_{\mathrm{C}}$,*

$$|z| \geq 0, \qquad |z| = 0 \iff z = 0_{\mathbb{C}}, \qquad |zw| = |z||w|.$$

Proof. The first two statements follow from the norm theorem and nonnegative root uniqueness. For multiplication,

$$|zw|^2 = N(zw) = N(z)N(w) = |z|^2|w|^2,$$

and both sides are nonnegative, so completed-root uniqueness gives the result. □

63.9 Complex inverse and division

Definition 63.22 (Complex apartness)**.** *A complex report z is apart from zero, written*

$$z \# 0_{\mathbb{C}},$$

when its norm is positively separated from zero:

$$N(z) \# 0_{\mathbb{R}}.$$

Definition 63.23 (Complex reciprocal)**.** *For $z \# 0_{\mathbb{C}}$, define*

$$z^{-1} := \frac{\overline{z}}{N(z)}.$$

In pair form,

$$(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right).$$

Theorem 63.24 (Complex inverse)**.** *For every $z \# 0_{\mathbb{C}}$,*

$$z z^{-1} = 1_{\mathbb{C}} = z^{-1} z.$$

Hence $\mathbb{C}_{\mathrm{C}}$ is a constructive field relative to complex apartness.

Proof. By definition,

$$z z^{-1} = \frac{z\overline{z}}{N(z)} = \frac{N(z)}{N(z)} = 1.$$

The denominator reciprocal exists by positive separation. □

Definition 63.25 (Complex division)**.** *For $w \# 0_{\mathbb{C}}$, define*

$$\frac{z}{w} := z w^{-1} = \frac{z\overline{w}}{N(w)}.$$

The denominator is therefore repaired by conjugate pairing and positive real norm closure. Complex division is not primitive multiplication by an unexplained symbol; it is compiled from conjugation, norm, completed-real reciprocal, and multiplication.

63.10 Rotation, phase, and circle reports

Definition 63.26 (Unit complex orientation)**.** *A report $u \in \mathbb{C}_{\mathrm{C}}$ is a* unit orientation *when*

$$N(u) = 1.$$

The unit orientations form the completed coordinate circle carrier

$$\mathbb{U}_{\mathrm{C}} := \{ u \in \mathbb{C}_{\mathrm{C}} : N(u) = 1 \}.$$

Theorem 63.27 (Unit orientations form a multiplicative group)**.** *Under complex multiplication, $\mathbb{U}_{\mathrm{C}}$ is an abelian group.*

Proof. Norm multiplicativity gives closure. The identity has norm one. If $N(u) = 1$, then $u^{-1} = \overline{u}$ and also has norm one. □

Definition 63.28 (Direction report)**.** *For $z\#0_{\mathbb{C}}$, define its unit direction report*

$$\operatorname{dir}(z) := \frac{z}{|z|}.$$

Then

$$N(\operatorname{dir}(z)) = 1.$$

A later angle coordinate may report this direction relative to the circle and π_{C} constructed in Section 61. The direction report is prior to any chosen branch of scalar argument.

Principle 63.29 (No global argument without a branch rule)**.** *The unit direction u is a completed circle report. A scalar angle* $\arg(u)$ *requires a selected cut, winding convention, and interval. Equal unit directions do not recover different winding histories.*

63.11 Complex order limitation

Theorem 63.30 (No compatible total ordered-field structure)**.** *There is no total order on $\mathbb{C}_{\mathrm{C}}$ extending the completed-real order and making every nonzero square positive in the ordered-field sense.*

Proof. If such an order existed, then $\mathbf{i}^2 = -1$ would have to be nonnegative because it is a square. But 1 is positive, so -1 is negative in any order extending the real order. Contradiction. □

Complex magnitude is ordered; complex orientation is not compressed into one compatible total scalar order.

63.12 Complex-report preservation theorem

Theorem 63.31 (Oriented-pair preservation)**.** *The pair report (a, b) is sufficient for complex field operations, conjugation, norm, and magnitude because those operations have been proved constant on the completed-real representative fibers. It is not sufficient for a property depending on:*

(i) the finite rational or Cauchy representatives of a and b;

(ii) source branches whose contributions produced the pair;

(iii) winding or path history;

(iv) quotient, boundary, or residual ledgers erased before pair formation;

(v) or the evaluation tree of a complex function.

Such a property descends to $\mathbb{C}_{\mathrm{C}}$ only after the projection-fiber criterion of Section 54 is proved.

64 Complex zero and complete operator zero

The symbol 0 now occurs in several constructed carriers. Their scalar reports may agree while their types and obligations differ. This section separates those zero predicates before any analytic function, functional equation, or zeta zero is discussed.

64.1 Zero is typed by carrier and obligation

Definition 64.1 (Scalar real zero)**.** *The completed-real zero is*

$$0_{\mathbb{R}} := [\underline{0}],$$

the Cauchy-equivalence class of the constant rational-zero family.

Definition 64.2 (Signed zero)**.** *A signed state is zero under integer or real signed equality when its positive and negative lanes balance. Its normalized report is* $0_{\mathbb{Z}}$ *or* $0_{\mathbb{R}}$*, while its cancellation ledger may remain nontrivial.*

Definition 64.3 (Pair zero)**.** *The oriented-pair zero is*

$$0_{\text{pair}} := (0_{\mathbb{R}}, 0_{\mathbb{R}}).$$

Definition 64.4 (Complex zero)**.** *The complex zero is the additive identity*

$$0_{\mathbb{C}} := 0_{\text{pair}}.$$

It is one admitted member of the complex carrier and not literal absence.

Pair zero and complex zero are the same carrier element under the complex pair grammar. They are named separately to distinguish the representational statement "both lanes report real zero" from the algebraic statement "this report is the complex additive identity."

Remark 64.5 (Many source reports, one complex zero)**.** *The following may all project to* $0_{\mathbb{C}}$*:*

$$(0,0), \qquad z + (-z), \qquad (a,b) + (-a,-b),$$

and any completed pair whose two component families converge to real zero. They need not have equal source or evaluation ledgers.

64.2 Evaluation ledgers

Definition 64.6 (Complex evaluation ledger)**.** *Let*

$$f : D \to \mathbb{C}_{\mathrm{C}}$$

be a complex-valued mapping. A declared evaluation ledger at $s \in D$ *is a report*

$$\Lambda_f(s) = (s, f, \mathcal{T}_s, \mathcal{S}_s, \mathcal{R}_s, \mathcal{B}_s, \mathcal{O}_s, \gamma_s, f(s)),$$

where, as relevant:

(i) $\mathcal{T}_s$ *is the finite or completed operation tree;*

(ii) $\mathcal{S}_s$ *is the family of source contributions and branch labels;*

(iii) $\mathcal{R}_s$ *is the residual or unresolved-obligation ledger;*

(iv) $\mathcal{B}_s$ *is the boundary or quotient ledger;*

(v) $\mathcal{O}_s$ *is the orientation, conjugation, reflection, or winding ledger;*

(vi) γ_s *is the construction or continuation history;*

(vii) and $f(s)$ *is the final complex output report.*

Not every application requires every coordinate. The signature must state which coordinates are relevant and operational.

Definition 64.7 (Complex output projection)**.** *Define*

$$\pi_{\mathbb{C}} : \Lambda_f(D) \to \mathbb{C}_{\mathrm{C}}, \qquad \pi_{\mathbb{C}}(\Lambda_f(s)) := f(s).$$

This projection forgets every ledger coordinate not encoded in the final oriented pair.

64.3 Residual and source closure

Definition 64.8 (Residual closure zero)**.** *A ledger has* residual closure zero*, written*

$$Z_{\mathrm{res}}(\Lambda) = 1,$$

when every residual obligation declared by the operation grammar has been discharged and the residual channel reports its typed zero

$$\mathcal{R} = 0_{\mathrm{res}}.$$

Definition 64.9 (Boundary closure zero)**.** *A ledger has* boundary closure zero *when every declared quotient, carry, exposed-boundary, or interface obligation is either exactly matched or retained in the typed closure state prescribed by the operation.*

Definition 64.10 (Source closure zero)**.** *A ledger has* source closure zero*, written*

$$Z_{\mathrm{src}}(\Lambda) = 1,$$

when every admitted source contribution is accounted for by the declared matching, cancellation, transport, or completion rule, with no unmatched source lane remaining.

Source closure does not mean the source ledger is empty. It means that its obligations close under the declared relation.

Example 64.11 (Scalar cancellation without ledger identity)**.** *Let* $z \neq 0_{\mathbb{C}}$*. The evaluation*

$$z + (-z) = 0_{\mathbb{C}}$$

has scalar output zero and may have exact source closure through an explicit opposing pairing. A primitive constant-zero evaluation also outputs $0_{\mathbb{C}}$ *but contains no such pair. The two ledgers share an output and differ in source history.*

Example 64.12 (Output zero with an unresolved auxiliary ledger)**.** *An operation may produce two complex branches that cancel in the final pair while retaining a nonzero declared remainder, boundary crossing, or unmet certificate obligation. Then*

$$\pi_{\mathbb{C}}(\Lambda) = 0_{\mathbb{C}}$$

but

$$Z_{\mathrm{res}}(\Lambda) = 0.$$

The scalar cancellation does not erase the unresolved ledger coordinate.

64.4 Root, kernel, and zero operator

Definition 64.13 (Root of a complex-valued mapping)**.** *A report $s \in D$ is a* complex root *of f when*

$$f(s) = 0_{\mathbb{C}}.$$

The root set is

$$Z(f) := \{s \in D : f(s) = 0_{\mathbb{C}}\}.$$

Definition 64.14 (Kernel)**.** *When f is a homomorphism, its kernel is*

$$\ker f := Z(f).$$

The word kernel *additionally records that the zero fiber is compatible with the algebraic operations preserved by f.*

Definition 64.15 (Zero operator)**.** *The* zero operator *from A to B is the mapping*

$$\mathbf{0}_{A\to B} : A \to B$$

satisfying

$$\mathbf{0}_{A\to B}(a) = 0_B$$

for every $a \in A$.

Theorem 64.16 (A root is not a zero operator)**.** *The statement*

$$f(s) = 0_{\mathbb{C}}$$

at one input does not imply

$$f = \mathbf{0}_{D\to\mathbb{C}_{\mathrm{C}}}.$$

Proof. The first statement concerns one input fiber. The second quantifies over every admissible input. A single root provides no such universal report. □

Definition 64.17 (Operator equality)**.** *Two operators $F, G : A \to B$ are extensionally equal when*

$$F(a) =_B G(a)$$

for every $a \in A$. They are constructively operator-equal only when the additional rule, domain, evaluation, and source structures declared relevant are also preserved.

The zero operator is therefore a typed mapping object, not merely the scalar zero of its codomain.

64.5 Complete ledger zero

Definition 64.18 (Zero signature)**.** *A* zero signature *is a declared finite family of zero obligations*

$$\Sigma = \{Z_{\mathbb{C}}, Z_{\mathrm{res}}, Z_{\mathrm{src}}, Z_{\mathrm{bdry}}, Z_{\mathrm{orient}}, \ldots\}$$

relevant to a class of evaluation ledgers.

Definition 64.19 (Complete ledger zero)**.** *A ledger* Λ *has* complete zero relative to Σ*, written*

$$Z_\Sigma(\Lambda) = 1,$$

when every zero obligation in Σ *is satisfied. In particular, when* $Z_{\mathbb{C}} \in \Sigma$*,*

$$\pi_{\mathbb{C}}(\Lambda) = 0_{\mathbb{C}}$$

is required but may not be sufficient.

There is no untyped universal ledger zero independent of a declared operation grammar. The signature states what must close.

Definition 64.20 (Complete operator zero)**.** *An operator* $F : A \to B$ *is a* complete operator zero relative to Σ *when, for every admissible input* $a \in A$*,*

$$Z_\Sigma(\Lambda_F(a)) = 1.$$

This is stronger than extensional equality with the scalar zero operator whenever Σ includes source, residual, boundary, orientation, or rule obligations not determined by the final codomain value.

64.6 The zero hierarchy

The principal zero reports now form the following typed hierarchy:

literal absence	:	no report exists,
zero token	:	one report object denotes no declared contribution,
signed zero	:	opposing magnitude lanes balance,
real zero	:	a completed-real equivalence class,
pair zero	:	$(0_{\mathbb{R}}, 0_{\mathbb{R}})$,
complex zero	:	the additive identity of $\mathbb{C}_{\mathrm{C}}$,
residual zero	:	all declared residual obligations close,
source zero	:	all declared source obligations are accounted for,
zero operator	:	every input receives codomain zero,
complete ledger zero	:	every obligation in Σ closes.

No upward implication is automatic merely because the displayed scalar symbol is 0.

64.7 Scalar zero is not automatically complete zero

Theorem 64.21 (No automatic zero upgrade)**.** *Let* $\pi_{\mathbb{C}} : \mathcal{L} \to \mathbb{C}_{\mathrm{C}}$ *be the complex output projection on a ledger carrier* $\mathcal{L}$*, and let* Z_Σ *be a stronger zero predicate. From*

$$\pi_{\mathbb{C}}(\Lambda) = 0_{\mathbb{C}}$$

it follows that $Z_\Sigma(\Lambda) = 1$ *only if* Z_Σ *is constant on the complex-output fiber over* $0_{\mathbb{C}}$*.*

Proof. If the predicate is not constant on the zero fiber, there exist ledgers Λ_1 and Λ_2 with

$$\pi_{\mathbb{C}}(\Lambda_1) = \pi_{\mathbb{C}}(\Lambda_2) = 0_{\mathbb{C}}$$

but

$$Z_\Sigma(\Lambda_1) \neq Z_\Sigma(\Lambda_2).$$

The common scalar output then cannot determine the stronger predicate. □

Theorem 64.22 (Zero-predicate factorization criterion)**.** *The following are equivalent:*

(i) there exists a predicate

$$\widehat{Z}_\Sigma : \mathbb{C}_{\mathrm{C}} \to \{0, 1\}$$

such that

$$Z_\Sigma = \widehat{Z}_\Sigma \circ \pi_{\mathbb{C}};$$

(ii) whenever two ledgers have the same complex output, they have the same Σ-zero status:

$$\pi_{\mathbb{C}}(\Lambda_1) = \pi_{\mathbb{C}}(\Lambda_2) \Longrightarrow Z_\Sigma(\Lambda_1) = Z_\Sigma(\Lambda_2).$$

Proof. This is the predicate specialization of the projection-fiber preservation theorem from Section 54. □

Corollary 64.23 (When complex zero is sufficient)**.** *If Z_Σ factors through $\pi_{\mathbb{C}}$ and*

$$\widehat{Z}_\Sigma(0_{\mathbb{C}}) = 1,$$

then

$$f(s) = 0_{\mathbb{C}} \Longrightarrow Z_\Sigma(\Lambda_f(s)) = 1.$$

Without the factorization theorem, the implication is unsupported.

64.8 Complete zero and construction equivalence

Theorem 64.24 (Complete zero need not imply ledger identity)**.** *Even when two ledgers both satisfy $Z_\Sigma = 1$, they need not be constructively equal. Complete zero classifies closure status relative to Σ; it does not necessarily identify source histories, operation trees, or certificates.*

Proof. Two distinct source pairings may each close every obligation in Σ. Unless Σ includes full source and rule identity, both are complete zero ledgers but remain distinct constructions. □

Thus complete zero is itself a projection unless its signature contains the entire declared ledger.

64.9 Zero-preserving maps

Definition 64.25 (Zero-preserving map)**.** *A mapping $F : A \to B$ is scalar-zero-preserving when*

$$a = 0_A \Longrightarrow F(a) = 0_B.$$

It is Σ-zero-preserving when complete zero of an input ledger implies complete zero of the output ledger under the translated signature.

Theorem 64.26 (Homomorphisms preserve scalar identity zero)**.** *Every additive homomorphism satisfies*

$$F(0_A) = 0_B.$$

Proof. Since $0_A + 0_A = 0_A$,

$$F(0_A) + F(0_A) = F(0_A).$$

Add the inverse of $F(0_A)$ to both sides to obtain $F(0_A) = 0_B$. □

This theorem concerns the carrier identity. Preservation of source, residual, or operator zero requires additional ledger theorems.

64.10 Application gate for complex analysis

Any later use of a statement

$$f(s) = 0_{\mathbb{C}}$$

must declare which of the following is intended:

(1) only the final longitudinal and transverse reports vanish;

(2) every finite or completed residual channel closes;

(3) reflected, conjugate, quotient, and source lanes are all accounted for;

(4) the evaluation is zero under a declared operator predicate;

(5) or the complete ledger satisfies a specified zero signature Σ.

Principle 64.27 (Complex-zero audit)**.** *A scalar complex root may be promoted to a stronger constructive zero only by one of two routes:*

(i) directly verify the additional zero obligations on the evaluation ledger; or

(ii) prove that the stronger predicate factors through the complex output projection.

No appeal to the written symbol 0 *can replace this obligation.*

64.11 Part XV summary

Sections 62–64 compile the prior manuscript into the following dependency chain:

$$\begin{aligned}&\text{finite and completed operations} \longrightarrow \text{typed carriers and closure}\\ &\longrightarrow \text{identities and inverses} \longrightarrow \text{groups, rings, and fields}\\ &\longrightarrow \text{congruences and quotient carriers}\\ &\longrightarrow \text{oriented pairs and } \mathbf{i}^2 = -1\\ &\longrightarrow \text{complex field operations}\\ &\longrightarrow \text{typed zero ledgers and fiber preservation.}\end{aligned}$$

The essential conclusions are:

(1) algebraic carriers are compiled report systems, not primitive universes;

(2) closure is relative to an operation, admissible domain, carrier, and equality;

(3) identities and inverses are typed and unique only under the declared equality;

(4) $\mathbb{N}_{\mathrm{C}}$, $\mathbb{Z}_{\mathrm{C}}$, $\mathbb{Q}_{\mathrm{C}}$, and $\mathbb{R}_{\mathrm{C}}$ support the expected additive and multiplicative structures through previously proved construction laws;

(5) constructive field inversion requires positive apartness from zero;

(6) quotient operations are valid exactly when the quotient relation is a congruence;

(7) operational closure and algebraic closure are different obligations;

(8) the complex unit is a transverse quarter-turn operator, not an unexplained primitive scalar;

(9) $\mathbb{C}_{\mathrm{C}}$ is the oriented-pair field and equivalently the quadratic quotient $\mathbb{R}_{\mathrm{C}}[T]/(T^2+1)$;

(10) conjugation is transverse reflection, norm is $z\overline{z}$, magnitude is its completed nonnegative root, and division is conjugate-norm repair;

(11) complex scalar zero, residual zero, source zero, zero operator, and complete ledger zero are different typed predicates;

(12) $f(s) = 0_{\mathbb{C}}$ certifies only complex output zero unless every stronger zero obligation is directly checked or proved constant on the complex-output fiber;

(13) and even complete zero relative to a signature need not erase distinct construction histories.

The next part should construct finite sequences, partial sums, series rules, and completed aggregation before continuity, differentiation, integration, exponential, logarithmic, gamma, and analytic continuation are introduced.

Part XVI
Analysis

65 Series and completed aggregation

Analysis begins only after finite arithmetic, signed orientation, completion, mappings, and equality preservation have already been constructed. A series is not primitive addition over an already completed infinity. It is first an indexed rule that produces finite partial-sum ledgers; a completed aggregate appears only after a convergence or summability criterion has been declared.

65.1 Finite partial sums

Definition 65.1 (Series term family)**.** *Let I be an indexed construction carrier and let A be an additive carrier. A **term family** is a mapping*

$$a : I \to A.$$

When I is a natural-number prefix carrier, write its values as

$$a_0, a_1, \ldots, a_n.$$

The family includes both the term reports and the rule or table by which the terms are generated.

Definition 65.2 (Finite partial-sum ledger)**.** *For a finite prefix $[0, n]$, the **finite partial-sum ledger** is*

$$S_n = (a_0, a_1, \ldots, a_n; +, \gamma_n)$$

whose scalar additive report is

$$s_n = \sum_{k=0}^{n} a_k.$$

The symbol $\sum$ abbreviates repeated source-preserving addition over the retained prefix. It does not erase the term indices, grouping rule, or addition order unless a later theorem proves that those data are irrelevant.

Principle 65.3 (Finite-first aggregation)**.** *Every series construction begins with finite partial sums. No completed aggregate may be inferred from the notation*

$$\sum_{k=0}^{\infty} a_k$$

without a declared completion rule for the partial-sum family.

Definition 65.4 (Partial-sum mapping)**.** *The partial-sum mapping of a term family is*

$$\Sigma_a : \mathbb{N}_{\mathrm{C}} \to A, \qquad \Sigma_a(n) = \sum_{k=0}^{n} a_k.$$

Thus a series rule is the composed construction

$$a_k \;\rightsquigarrow\; S_n \;\rightsquigarrow\; \Sigma_a(n).$$

65.2 Convergence of partial-sum ledgers

Definition 65.5 (Convergent series)**.** *Let A be a completed metric or ordered additive carrier. The series $\sum a_k$ **converges** to L when the partial-sum family (s_n) converges to L under the limit grammar of Section 55:*

$$\lim_{n\to\infty} s_n = L.$$

The completed aggregate is the limit class of the partial-sum ledger family, not an additional primitive act of adding infinitely many objects at once.

Definition 65.6 (Series residual)**.** *If $\sum a_k$ converges to L, the stage-n residual is the signed completed report*

$$R_n = L - s_n.$$

A convergence proof supplies, for every positive resolution ε, a stage N such that

$$|R_n| < \varepsilon$$

for every $n \geq N$.

Theorem 65.7 (Series convergence is partial-sum convergence)**.** *A series has a completed sum in this manuscript exactly when its partial-sum mapping has a completed limit in the declared completion carrier.*

Proof. The term family alone produces only finite prefixes and their partial sums. A completed value can be assigned only by applying the previously constructed limit operation to the partial-sum family. Therefore the completed series sum is precisely the limit of the partial sums. Conversely, any convergent partial-sum family supplies the completed aggregate by definition. □

65.3 Absolute convergence

Definition 65.8 (Absolute series)**.** *For terms in a signed or complex completed carrier, define the magnitude term family*

$$b_k = |a_k|.$$

The original series is ***absolutely convergent*** *when*

$$\sum_{k=0}^{\infty} b_k$$

converges.

Theorem 65.9 (Absolute convergence implies Cauchy convergence)**.** *In a complete normed additive carrier, if $\sum |a_k|$ converges, then $\sum a_k$ converges.*

Proof. For $m > n$, the signed tail satisfies

$$\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|.$$

If the magnitude series converges, its tails are eventually smaller than every positive resolution. Hence the original partial sums are Cauchy. Completeness supplies a completed limit. □

Principle 65.10 (Magnitude domination is a ledger theorem)**.** *Absolute convergence is not an aesthetic preference for positive terms. It is a constructive domination theorem: every signed or oriented tail is bounded by a positive tail whose completed disappearance has been proved.*

65.4 Conditional convergence

Definition 65.11 (Conditional convergence)**.** *A series is* ***conditionally convergent*** *when*

$$\sum a_k$$

converges but

$$\sum |a_k|$$

does not converge. Equivalently, signed cancellation is necessary for the completed aggregate.

Remark 65.12 (Why conditional convergence preserves more history)**.** *For an absolutely convergent series, large signed rearrangements are dominated by a convergent positive magnitude ledger. For a conditionally convergent series, cancellation order may contribute to the completed value. Therefore order and grouping cannot be casually quotiented away.*

65.5 Rearrangement

Definition 65.13 (Rearranged term family)**.** *Let $\rho : \mathbb{N}_{\mathrm{C}} \to \mathbb{N}_{\mathrm{C}}$ be a bijective reindexing. The rearranged family is*

$$a_k^{\rho} = a_{\rho(k)}.$$

Its partial sums are

$$s_n^{\rho} = \sum_{k=0}^{n} a_{\rho(k)}.$$

The rearrangement ledger retains ρ.

Theorem 65.14 (Absolute rearrangement preservation)**.** *If $\sum a_k$ is absolutely convergent, every bijective rearrangement has the same completed sum.*

Proof. Given a positive resolution, choose a finite prefix after which the absolute tail is smaller than the resolution. Every rearranged finite prefix eventually contains all terms from the original large finite core; the remaining discrepancy is contained in the absolutely small tail. Thus rearranged and original partial sums become arbitrarily close to the same limit. □

Principle 65.15 (Rearrangement audit)**.** *A rearranged infinite aggregation is valid only after the property under discussion is proved constant over the rearrangement fiber. Absolute convergence supplies such a theorem for the completed scalar sum. Conditional convergence does not supply it automatically.*

65.6 Source preservation

Definition 65.16 (Source-preserving series ledger)**.** *A source-preserving series ledger is*

$$\Lambda_{\sum a} = (a, \Sigma_a, (s_n), \gamma, +, \mathcal{C}),$$

where a is the term rule, Σ_a is the partial-sum mapping, (s_n) is the partial-sum family, γ records grouping and order, $+$ is the additive carrier operation, and $\mathcal{C}$ is the convergence certificate.

Theorem 65.17 (Completed scalar sum does not recover the series ledger)**.** *Two series may have the same completed sum while having different term rules, partial-sum paths, convergence rates, cancellation histories, or rearrangement ledgers.*

Proof. Let one series be $1 + (-1) + 0 + 0 + \cdots$ and another be the constant zero series. Both may project to completed sum 0, but the first has a nontrivial finite cancellation ledger and the second does not. Equality of completed scalar sums therefore does not imply equality of series ledgers. □

66 Continuity

Continuity is a property of a completed mapping between completed carriers. It is not inherited from a visual curve, a coordinate drawing, or an assumed continuum. It must be stated through input and output relations.

66.1 Input-output completion

Definition 66.1 (Completed mapping)**.** *Let X and Y be completed carriers and let*

$$f : X \to Y$$

be a mapping. An evaluation of f at x has an evaluation ledger

$$\Lambda_f(x) = (x, f, x \mapsto f(x), \gamma_x).$$

Continuity concerns the behavior of these output reports as input reports vary inside the completed carrier.

Definition 66.2 (Continuity at a point)**.** *Let X and Y carry completed distance or resolution relations. The mapping $f : X \to Y$ is **continuous at** x_0 when for every positive output resolution ε there exists a positive input resolution δ such that*

$$d_X(x, x_0) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(x_0)) < \varepsilon.$$

This is a relation between completed input differences and completed output differences.

Definition 66.3 (Sequential continuity)**.** *The mapping f is sequentially continuous at x_0 when every convergent input family*

$$x_n \to x_0$$

produces a convergent output family

$$f(x_n) \to f(x_0).$$

Theorem 66.4 (Metric equivalence of the two continuity reports)**.** *For mappings between completed metric carriers, epsilon–delta continuity at x_0 is equivalent to sequential continuity at x_0.*

Proof. If epsilon–delta continuity holds and $x_n \to x_0$, choose the corresponding δ for a given ε. Eventually $d_X(x_n, x_0) < \delta$, so eventually $d_Y(f(x_n), f(x_0)) < \varepsilon$.

Conversely, if epsilon–delta continuity fails, then some $\varepsilon_0 > 0$ has the property that for every positive δ there is an x with $d_X(x, x_0) < \delta$ but $d_Y(f(x), f(x_0)) \geq \varepsilon_0$. Choosing such an x_n for $\delta = 1/n$ constructs a sequence converging to x_0 whose output fails to converge to $f(x_0)$. □

66.2 Continuity and projection fibers

Principle 66.5 (Continuity is not source preservation)**.** *Continuity of the scalar or coordinate output of f does not imply preservation of every evaluation ledger coordinate. To promote continuity to a stronger source statement, the stronger predicate must factor through the completed output relation.*

Definition 66.6 (Ledger-continuity)**.** *Let $\Lambda_f(x)$ be a selected evaluation ledger. The map f is **ledger-continuous** for a ledger coordinate P when*

$$x_n \to x_0 \quad \Longrightarrow \quad P(\Lambda_f(x_n)) \to P(\Lambda_f(x_0))$$

inside the declared carrier for that ledger coordinate.

Remark 66.7. *Ordinary continuity usually concerns final values. Primitive Constructive Logic permits that convention, but it does not allow a final-value continuity theorem to be silently used as a source-ledger continuity theorem.*

66.3 Uniform continuity

Definition 66.8 (Uniform continuity)**.** *A mapping $f : X \to Y$ is **uniformly continuous** on a domain $D \subseteq X$ when for every positive output resolution ε there exists one positive input resolution δ such that for all $x, y \in D$,*

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon.$$

The same δ works throughout the declared domain.

Theorem 66.9 (Continuous image of a convergent family)**.** *If f is continuous at x_0 and $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

Proof. This is the forward implication in the metric equivalence theorem. It is recorded separately because it is the continuity rule most often used in later analysis. □

66.4 Composition

Theorem 66.10 (Composition of continuous mappings)**.** *If $f : X \to Y$ is continuous at x_0 and $g : Y \to Z$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. Given an output resolution for Z, continuity of g supplies a resolution in Y. Continuity of f supplies a resolution in X that forces the Y-output inside that resolution. The composed implication is exactly continuity of $g \circ f$. □

67 Differentiation

Differentiation is the completed report of signed ratio states produced by changing an input and comparing the resulting output. No derivative is primitive. Every finite difference quotient remains a finite ratio object; the derivative appears only after a limiting operation is applied to the family of such quotients.

67.1 Finite difference quotient

Definition 67.1 (Input increment)**.** *Let $f : D \subseteq \mathbb{R}_{\mathrm{C}} \to \mathbb{R}_{\mathrm{C}}$ and let x be a point of D. A nonzero signed input increment is a report h such that $h \neq 0$ and $x + h \in D$.*

Definition 67.2 (Finite difference quotient)**.** *The **finite difference quotient** of f at x with increment h is the signed ratio state*

$$Q_f(x; h) = \frac{f(x + h) - f(x)}{h}.$$

Its complete ledger retains

$$(x, h, x + h, f(x), f(x + h), f(x + h) - f(x), h, \textit{division rule}).$$

Principle 67.3 (Finite quotients are not derivatives)**.** *For $h \neq 0$, $Q_f(x; h)$ is a finite completed ratio. It is not yet the derivative. The derivative is a later limit report of the family $h \mapsto Q_f(x; h)$ as h approaches zero through nonzero increments.*

67.2 Derivative as a limit report

Definition 67.4 (Derivative)**.** *The function f is **differentiable at** x with derivative L when*

$$\lim_{h \to 0,\ h \neq 0} Q_f(x; h) = L.$$

Write

$$f'(x) = L.$$

The derivative ledger retains the difference-quotient family and the convergence certificate.

Definition 67.5 (Residual form of differentiability)**.** *Equivalently, f is differentiable at x with derivative L when there exists a residual family $\eta(h)$ such that*

$$f(x + h) = f(x) + Lh + h\eta(h)$$

with

$$\eta(h) \to 0.$$

This separates the linear carrier Lh from the vanishing residual report.

Theorem 67.6 (Differentiability implies continuity)**.** *If f is differentiable at x, then f is continuous at x.*

Proof. By the residual form,

$$f(x+h) - f(x) = Lh + h\eta(h).$$

As $h \to 0$, both Lh and $h\eta(h)$ converge to zero, since L is fixed and $\eta(h)$ remains eventually bounded while tending to zero. Hence $f(x+h) \to f(x)$. □

67.3 Basic derivative laws

Theorem 67.7 (Linearity of derivative)**.** *If f and g are differentiable at x, then $f+g$ is differentiable at x and*

$$(f+g)'(x) = f'(x) + g'(x).$$

For a scalar report c,

$$(cf)'(x) = cf'(x).$$

Proof. The finite difference quotient of $f+g$ is the sum of the finite difference quotients of f and g. Taking limits and using algebraic preservation of limits gives the result. Scalar multiplication is identical. □

Theorem 67.8 (Product rule)**.** *If f and g are differentiable at x, then fg is differentiable at x and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. For nonzero h,

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}.$$

The difference quotients converge to $f'(x)$ and $g'(x)$, and differentiability implies $g(x+h) \to g(x)$. Limit preservation gives the stated formula. □

Theorem 67.9 (Chain rule)**.** *If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Proof. The input change h produces an intermediate change

$$k(h) = f(x+h) - f(x).$$

Differentiability of f gives $k(h) = f'(x)h + h\eta(h)$, so $k(h) \to 0$. Differentiability of g at $f(x)$ gives

$$g(f(x)+k) - g(f(x)) = g'(f(x))k + k\theta(k)$$

with $\theta(k) \to 0$. Substitute $k = k(h)$ and divide by h. The residual terms vanish, leaving $g'(f(x))f'(x)$. □

67.4 Complex differentiability

Definition 67.10 (Complex difference quotient)**.** *For $f : D \subseteq \mathbb{C}_{\mathrm{C}} \to \mathbb{C}_{\mathrm{C}}$, the finite complex difference quotient is*

$$Q_f(z;h) = \frac{f(z+h) - f(z)}{h}, \qquad h \in \mathbb{C}_{\mathrm{C}},\ h \neq 0.$$

The denominator may approach zero through many orientation lanes.

Definition 67.11 (Complex derivative)**.** *The complex derivative $f'(z)$ exists when $Q_f(z;h)$ has the same completed limit for every nonzero complex approach $h \to 0$ admitted by the domain.*

Principle 67.12 (Complex differentiability is stronger than two real directional derivatives)**.** *A complex derivative requires approach-lane invariance. Agreement along only the longitudinal and transverse axes is evidence, not the full theorem.*

68 Integration

Integration is completed accumulation over partitioned domains. It must retain finite partitions, area-bearing cells, refinement, lower and upper accumulation, and the completed integral report.

68.1 Finite partitions and cells

Definition 68.1 (Interval partition)**.** *For a completed real interval $[a, b]$, a finite partition is an ordered report*

$$P = (x_0, x_1, \ldots, x_n)$$

with

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Each subinterval $[x_{j-1}, x_j]$ is a whole cell in the partition ledger.

Definition 68.2 (Mesh)**.** *The mesh of P is*

$$\|P\| = \max_{1 \leq j \leq n} (x_j - x_{j-1}).$$

The mesh reports the largest retained cell width.

Definition 68.3 (Tagged Riemann ledger)**.** *A tagged partition includes tags $t_j \in [x_{j-1}, x_j]$. Its finite accumulation report is*

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The ledger retains every cell, tag, width, height report, and product cell.

68.2 Lower and upper accumulation

Definition 68.4 (Lower and upper cell reports)**.** *For a bounded function $f : [a, b] \to \mathbb{R}_{\mathrm{C}}$ and subinterval $I_j = [x_{j-1}, x_j]$, define*

$$m_j = \inf_{x \in I_j} f(x), \qquad M_j = \sup_{x \in I_j} f(x),$$

when these infimum and supremum reports exist.

Definition 68.5 (Lower and upper sums)**.** *The lower and upper sums are*

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}),$$

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

They are finite accumulation ledgers whose cell values come from lower and upper bounds.

Definition 68.6 (Riemann integrability)**.** *A bounded function f is Riemann integrable on $[a, b]$ when the lower and upper accumulations close to the same completed report:*

$$\sup_P L(f, P) = \inf_P U(f, P).$$

The common completed value is

$$\int_a^b f(x)\, dx.$$

Principle 68.7 (Integral as completed squeeze)**.** *An integral is not a primitive area value. It is the completed equality of lower and upper finite accumulation ledgers under refinement.*

68.3 Area-bearing interpretation

Remark 68.8 (Why integration waits for measure)**.** *The product $f(t_j)(x_j - x_{j-1})$ is a coordinate report of an accumulation cell. The geometric legitimacy of treating such products as area depends on the already constructed measure and coordinate-projection sections. The integral does not smuggle area into the foundation before closure and measure exist.*

68.4 Refinement

Definition 68.9 (Partition refinement)**.** *A partition Q refines P when every partition point of P also occurs in Q. Refinement subdivides cells into smaller whole cells; it does not create fractional indices.*

Theorem 68.10 (Refinement monotonicity)**.** *If Q refines P, then*

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P).$$

Proof. On each original cell, subdivision can only raise the sum of local infimum-width products and can only lower the sum of local supremum-width products. Summing across all original cells gives the inequalities. □

68.5 Fundamental theorem as a preservation result

Theorem 68.11 (Integral of a derivative)**.** *If F is continuously differentiable on $[a, b]$, then*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Proof sketch. For each partition cell, the mean-value relation or a uniform continuity argument compares

$$F(x_j) - F(x_{j-1})$$

with

$$F'(t_j)(x_j - x_{j-1}).$$

The finite telescoping sum of the exact differences is $F(b) - F(a)$. Uniform continuity of F' forces the total error between the derivative accumulation ledger and the telescoping ledger to vanish under refinement. The completed reports are therefore equal. □

Principle 68.12 (Integration preserves declared accumulation, not every source)**.** *Equality of completed integrals does not by itself identify integrands, partitions, geometric cells, or physical accumulation histories. Any stronger identification requires a fiber-preservation theorem.*

69 Exponential, logarithmic, and gamma reports

The zeta compiler requires powers n^{-s}, exponential decay, logarithmic derivatives, and the gamma factor. These objects must be constructed from series, products, integrals, and analytic continuation rather than assumed as primitive functions.

69.1 Exponential report

Definition 69.1 (Exponential series)**.** *For $z \in \mathbb{C}_{\mathrm{C}}$, define the exponential report by the absolutely convergent series*

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Here z^k is the finite power ledger, $k!$ is the finite product of positive integer reports, and division by $k!$ is a completed ratio in the complex carrier.

Theorem 69.2 (Absolute convergence of the exponential series)**.** *For every completed complex report z, the exponential series converges absolutely.*

Proof. The magnitude terms are $|z|^k/k!$. For sufficiently large k, the ratio of consecutive magnitude terms is

$$\frac{|z|^{k+1}/(k+1)!}{|z|^k/k!} = \frac{|z|}{k+1} < \frac{1}{2}.$$

The tail is therefore dominated by a convergent geometric tail. □

Theorem 69.3 (Exponential addition law)**.** *For all complex reports z, w,*

$$\exp(z + w) = \exp(z)\exp(w).$$

Proof sketch. Multiply the two absolutely convergent series. Absolute convergence justifies regrouping by total degree. The coefficient of degree n becomes

$$\sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \frac{(z+w)^n}{n!}$$

by the finite binomial theorem. The completed product series is therefore the exponential series of $z + w$. □

69.2 Logarithmic reports

Definition 69.4 (Local logarithm near one)**.** *For $|u| < 1$, define*

$$\log(1+u) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{k}.$$

This is a convergent series report in the completed complex carrier.

Definition 69.5 (Positive real logarithm)**.** *For a positive real report $x > 0$, $\log x$ is the completed inverse of the positive exponential mapping:*

$$\exp(y) = x, \qquad y \in \mathbb{R}_{\mathrm{C}}.$$

Existence and uniqueness follow from continuity, monotonicity, and completeness of the positive real exponential report.

Definition 69.6 (Complex logarithm branch)**.** *For $z \neq 0$, a complex logarithm report consists of a magnitude lane and an orientation lane:*

$$\mathrm{Log}_{\Gamma}(z) = \log|z| + \mathbf{i}\,\mathrm{Arg}_{\Gamma}(z),$$

where Γ is a declared branch or winding ledger. Different branch ledgers may report values differing by full turn periods.

Principle 69.7 (Logarithm is branch-typed)**.** *The symbol $\log z$ is incomplete for complex z until a branch, cut, or winding convention is declared. Projection to a principal value is a quotient of orientation history.*

69.3 Complex powers

Definition 69.8 (Complex power by exponential-logarithm compilation)**.** *For $z \neq 0$ and complex exponent s, define the branch-typed power*

$$z_{\Gamma}^{s} = \exp(s\,\mathrm{Log}_{\Gamma}(z)).$$

When z is a positive real report, the principal real logarithm supplies the standard branch and the subscript may be omitted.

Remark 69.9. *This is the analytic continuation of finite operational powers into a completed exponential-logarithm grammar. It does not retroactively redefine finite powers as primitive exponentials.*

69.4 Gamma report

Definition 69.10 (Gamma integral)**.** *For real $x > 0$, define*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

as an improper integral report: first integrate over finite intervals $[\varepsilon, R]$, then take the completed limit as $\varepsilon \to 0^+$ and $R \to \infty$ when the two boundary obligations close.

Definition 69.11 (Complex gamma on the right half-plane)**.** *For complex s with $\Re(s) > 0$, define*

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt,$$

where

$$t^{s-1} = \exp((s-1)\log t)$$

uses the positive real logarithm of t.

Theorem 69.12 (Gamma recurrence)**.** *For $\Re(s) > 0$,*

$$\Gamma(s+1) = s\Gamma(s).$$

Proof sketch. On finite intervals, integrate by parts:

$$\int t^s e^{-t}\, dt = -t^s e^{-t} + s \int t^{s-1} e^{-t}\, dt.$$

The boundary terms vanish under the improper-integral completion for $\Re(s) > 0$. The completed equality follows. □

Definition 69.13 (Meromorphic gamma continuation)**.** *The recurrence*

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

extends the gamma report stepwise to complex reports not equal to nonpositive integers. The excluded reports are pole obligations where the denominator lane is not apart from zero.

70 Analytic continuation

Analytic continuation is not permission to replace one function with another merely because symbols match. It is a domain-extension construction with a preservation theorem.

70.1 Original report and extension target

Definition 70.1 (Analytic germ)**.** *An **analytic germ** at a point or connected domain consists of a local power-series rule, its convergence neighborhood, and its evaluation ledger. Write*

$$\mathcal{G} = (U, f, \Lambda_f).$$

Definition 70.2 (Continuation step)**.** *A continuation step from (U, f) to (V, g) is admissible when $U \cap V$ is nonempty and*

$$f(z) = g(z)$$

for every z in a nontrivial connected overlap where both sides are defined, with equality understood as completed scalar equality unless a stronger ledger equality is explicitly proved.

Definition 70.3 (Analytic continuation ledger)**.** *An analytic continuation ledger from an initial germ $\mathcal{G}_0$ to a target point z is a finite or coherently completed chain*

$$(U_0, f_0) \rightsquigarrow (U_1, f_1) \rightsquigarrow \cdots \rightsquigarrow (U_m, f_m)$$

with $z \in U_m$ and equality on every overlap. The ledger retains the domains, overlap witnesses, identity theorem used, and any singularities or branch choices crossed or avoided.

70.2 Identity theorem

Theorem 70.4 (Identity theorem)**.** *Let U be a connected complex domain. If analytic functions $f, g : U \to \mathbb{C}_{\mathrm{C}}$ agree on a set with an accumulation point in U, then they agree throughout U.*

Proof sketch. At an accumulation point, all coefficients of the local power series of $f - g$ vanish, so $f - g$ vanishes on a neighborhood. The set on which $f - g$ vanishes is both open and closed inside the connected continuation domain, hence it is all of U. □

Principle 70.5 (What analytic continuation preserves)**.** *Analytic continuation preserves the scalar analytic function specified by the identity theorem. It does not automatically preserve source formula, term-by-term convergence, product expansion, branch history, integration path, or operator-zero ledger unless those are separately carried and shown invariant.*

70.3 Branching and monodromy

Definition 70.6 (Branch ledger)**.** *A branch ledger records the path, cuts, winding counts, and singularity avoidances used to continue a multivalued analytic report such as a logarithm or power.*

Definition 70.7 (Monodromy report)**.** *If continuation along two paths with the same endpoints gives different branch values, the discrepancy is the monodromy report. It is not contradiction; it is retained orientation and winding history.*

Principle 70.8 (Single-valuedness obligation)**.** *A continued report may be treated as a single-valued function on a domain only after the continuation ledger proves path-independence on that domain or a branch selection is declared.*

70.4 Continuation versus original series

Definition 70.9 (Series domain)**.** *The series domain of a power series or Dirichlet series is the region where its defining partial sums converge to the reported value.*

Definition 70.10 (Continued domain)**.** *The continued domain is the region reached by analytic continuation of the original germ, minus declared singularities and branch exclusions.*

Principle 70.11 (No termwise inheritance outside convergence)**.** *Outside the series domain, the continued value is not the value of the original divergent termwise series. It is the value of the continued analytic report.*

70.5 Zeros under continuation

Definition 70.12 (Continued zero)**.** *A continued zero of an analytically continued report F at z is a point where*

$$F(z) = 0_{\mathbb{C}}.$$

The zero ledger includes the continuation path or domain certificate used to define $F(z)$.

Principle 70.13 (Continuation-zero audit)**.** *A zero obtained after analytic continuation certifies complex scalar zero of the continued report. It certifies complete source or operator zero only if the stronger zero predicate is proved invariant under the continuation chain and constant on the complex-output fiber.*

70.6 Analysis summary

Sections 65–70 construct the analysis layer required before any zeta application:

$$\begin{gathered}\text{finite term families} \longrightarrow \text{partial-sum ledgers} \longrightarrow \text{series convergence} \longrightarrow \text{completed aggregation},\\ \text{completed input-output relations} \longrightarrow \text{continuity},\\ \text{signed finite difference quotients} \longrightarrow \text{derivatives as completed quotient limits},\\ \text{finite partitions and lower/upper accumulation} \longrightarrow \text{completed integrals},\\ \text{series and integral compilers} \longrightarrow \exp, \log, \Gamma,\\ \text{local analytic germs and overlap identities} \longrightarrow \text{analytic continuation}.\end{gathered}$$

The essential conclusions are:

(1) a series is completed only through convergence of its partial-sum ledgers;

(2) absolute convergence supplies a rearrangement-preservation theorem, while conditional convergence retains order-sensitive cancellation history;

(3) continuity is a property of completed input-output relations, not a drawing of a curve;

(4) a derivative is a limit of signed ratio states, not a finite quotient;

(5) integration is completed lower/upper accumulation over finite area-bearing partition ledgers;

(6) exponentials, logarithms, complex powers, and gamma reports are compiled analytic objects with branch and boundary obligations;

(7) analytic continuation preserves the scalar analytic report supplied by the identity theorem, not every source construction by default;

(8) branch ledgers and monodromy record path-sensitive continuation history;

(9) and zeros of continued functions require the same complex-zero audit established in Section 64.

The next part can construct the zeta report itself: finite Dirichlet partial sums, the half-plane of convergence, Euler product where admissible, completed continuation, reflection, and the exact zero-predicate audit required before any critical-half theorem is stated.

Part XVII
Zeta, Indexed Zero, and Critical-Half Reconstruction

71 Construction of the zeta report

71.1 Finite Dirichlet stages

For a completed complex report $s \in \mathbb{C}_{\mathrm{C}}$ and a positive finite stage N, define the finite Dirichlet ledger

$$\mathcal{D}_N(s) = \left[(1, 1^{-s}), (2, 2^{-s}), \ldots, (N, N^{-s})\right]$$

and its scalar aggregation

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-s}.$$

The source index n is part of each term. Equal complex values arising at different indices remain different occurrences until a declared projection identifies them.

Definition 71.1 (Indexed zeta stage report)**.** *The stage report is*

$$Z_N(s) = (s, N, \mathcal{D}_N(s), \zeta_N(s), \tau_N, \partial_N),$$

where τ_N records the ordered aggregation route and ∂_N records the uncompiled tail obligation.

The scalar value $\zeta_N(s)$ is one coordinate of $Z_N(s)$; it is not the complete report occurrence.

71.2 The convergent half-plane

Write $s = \sigma + it$. For $\sigma > 1$,

$$|n^{-s}| = n^{-\sigma},$$

and the positive comparison series $\sum n^{-\sigma}$ converges. Hence the Dirichlet stages form a Cauchy family uniformly on every closed half-plane $\sigma \geq 1 + \delta$ with $\delta > 0$.

Definition 71.2 (Dirichlet zeta report)**.** *For $\Re(s) > 1$, the zeta report is the completed family*

$$\zeta_{\mathrm{C}}(s) = \mathrm{Comp}\big(Z_N(s)\big)_{N \geq 1}.$$

Its scalar coordinate is the classical completed sum, while its report identity retains at least the input index s, the defining family, the source indices, and the completion rule.

Thus two occurrences may be scalar-equal without becoming occurrence-identical:

$$\pi_{\mathbb{C}}\big(\zeta_{\mathrm{C}}(s_1)\big) = \pi_{\mathbb{C}}\big(\zeta_{\mathrm{C}}(s_2)\big) \not\Rightarrow \zeta_{\mathrm{C}}(s_1) =_{\mathrm{occ}} \zeta_{\mathrm{C}}(s_2).$$

71.3 Finite Euler products

For a finite prime list $P_M = (p_1, \dots, p_M)$, define

$$\mathcal{E}_M(s) = \prod_{j=1}^{M} (1 - p_j^{-s})^{-1}.$$

Each factor is an indexed geometric-series report. Expanding the finite product gives one contribution for every positive integer whose prime support lies in P_M, with its prime-exponent ledger retained.

Theorem 71.3 (Euler compilation where absolutely convergent)**.** *For $\Re(s) > 1$, the completed finite-product family and the completed Dirichlet family have the same scalar coordinate:*

$$\pi_{\mathbb{C}}(\zeta_{\mathrm{C}}(s)) = \prod_{p} (1 - p^{-s})^{-1}.$$

The equality follows from unique finite prime factorization and absolute convergence, which permits the required reindexing and regrouping.

This theorem establishes scalar and source-compatible agreement on the original half-plane. It does not make the Dirichlet ledger and Euler ledger literally the same construction history.

71.4 Continuation beyond the source half-plane

Section 70 constructed analytic continuation as a chain of overlapping analytic reports whose values agree on overlap. Let

$$\mathfrak{Z} = (U_0, f_0; U_1, f_1; \dots)$$

be a continuation ledger with

$$U_0 = \{s : \Re(s) > 1\}, \qquad f_0(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The meromorphic continuation is the completed report generated by this ledger, with its simple pole at $s = 1$ retained as a typed singularity.

Definition 71.4 (Continued zeta occurrence)**.** *At every admissible $s \neq 1$, write*

$$\zeta_{\mathrm{C}}^{[s,\mathfrak{Z}]} = (s, \mathfrak{Z}, \pi_{\mathbb{C}}(\zeta(s)), \gamma_s),$$

where γ_s records the selected continuation route or its proved route-equivalence class.

The superscript is not decorative. It marks the occurrence that the bare notation $\zeta(s)$ abbreviates.

72 Completed zeta and reflection

72.1 Gamma and power factors

Using the gamma and complex-power reports constructed in Part XVI, define

$$\Lambda(s) = \pi_{\mathrm{C}}^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta_{\mathrm{C}}(s),$$

and

$$\xi(s) = \frac{1}{2} s(s-1) \Lambda(s).$$

The polynomial factor closes the pole reports at $s = 0$ and $s = 1$, yielding an entire scalar report.

The complete indexed occurrence is written

$$\Xi[s] = (s, \zeta_{\mathrm{C}}^{[s,\mathfrak{Z}]}, \Gamma[s/2], \pi_{\mathrm{C}}^{[-s/2]}, s(s-1)/2, \pi_{\mathbb{C}}(\xi(s)), \tau_s, \partial_s).$$

72.2 Theta-source construction

Let

$$\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

Poisson symmetry gives

$$\vartheta(x) = x^{-1/2}\vartheta(1/x).$$

After splitting the Mellin construction at $x = 1$ and transporting the lower interval by $x \mapsto 1/x$, one obtains a symmetric source representation of ξ whose paired power lanes are exchanged by $s \mapsto 1 - s$.

At each positive source node $x > 1$, retain the packet

$$\mathcal{P}_x[s] = \big(x, s, 1 - s, x^{s/2}, x^{(1-s)/2}, \iota_x, \Delta_x, \tau_x\big),$$

where ι_x is the declared reflection/inverse lineage and Δ_x is the radial imbalance defined in Section 74.

72.3 Functional reflection

Theorem 72.1 (Completed reflection)**.** *The completed scalar report satisfies*

$$\pi_{\mathbb{C}}(\Xi[s]) = \pi_{\mathbb{C}}(\Xi[1 - s]).$$

At the indexed-report level, reflection supplies a structure-preserving transport

$$\mathfrak{R} : \Xi[s] \longrightarrow \Xi[1 - s]$$

that exchanges the paired source lanes, reverses the horizontal imbalance, and retains the reflection genealogy.

Scalar equality under reflection is therefore the coordinate shadow of a typed transport, not an erasure of the two indexed occurrences.

72.4 Conjugation

Because the defining coefficients and integration kernels are real,

$$\pi_{\mathbb{C}}(\Xi[\overline{s}]) = \overline{\pi_{\mathbb{C}}(\Xi[s])}.$$

Conjugation reverses the transverse orientation while retaining the horizontal source coordinate. Together, reflection and conjugation generate the familiar fourfold scalar symmetry, but the four reports retain different input indices unless fixed by the corresponding transformations.

73 Indexed zero-predicate audit

73.1 A zero glyph never appears without a report position

Earlier sections established that every report token occupies a whole report position. Consequently, a written zero in an evaluation is not merely the carrier element $0_{\mathbb{C}}$ considered without occurrence. It is an indexed zero occurrence.

Definition 73.1 (Indexed complex-zero occurrence)**.** *For the completed zeta evaluation at s, define*

$$0_{\xi,s} = (0_{\mathbb{C}}; \xi, s, \mathfrak{Z}, \tau_s, \partial_s, \iota_s).$$

The fields after the semicolon are the occurrence index. They identify which construction reports the complex additive identity.

Thus

$$0_{\xi,s_1} =_{\text{value}} 0_{\xi,s_2} = 0_{\mathbb{C}}$$

does not imply

$$0_{\xi,s_1} =_{\text{occ}} 0_{\xi,s_2}.$$

73.2 Bare scalar notation is an abbreviation

Within PCL, the conventional statement

$$\xi(s) = 0_{\mathbb{C}}$$

is parsed as

$$\Xi[s] =_{\text{report}} 0_{\xi,s},$$

not as the context-free identification of an unindexed value with literal absence. The input s, the operator ξ, and the evaluation occurrence are already supplied by the syntax of the equation.

Principle 73.2 (No index deletion by typographical suppression)**.** *Omitting an index from written notation does not erase the report position that makes the expression meaningful. A suppressed index may be restored from the operator, argument, carrier, and construction context.*

Therefore the equation at s and the equation at t contain distinct zero occurrences even when both project to the same complex additive identity.

73.3 Value projection and report projection

There are two different maps:

$$\pi_{\mathbb{C}} : \Xi[s] \longmapsto \pi_{\mathbb{C}}(\xi(s)),$$

and

$$r_{\#} : \Xi[s] \longmapsto (\pi_{\mathbb{C}}(\xi(s)); \xi, s, \mathfrak{Z}, \tau_s, \partial_s, \iota_s).$$

The first deliberately forgets occurrence identity. The second is the PCL report map.

Theorem 73.3 (Indexed-report injectivity)**.** *Let the index of $r_{\#}$ retain every field declared relevant to the equality of completed zeta evaluations. Then*

$$r_{\#}(\Xi[s_1]) =_{\mathrm{C}} r_{\#}(\Xi[s_2]) \Longrightarrow \Xi[s_1] =_{\mathrm{C}} \Xi[s_2].$$

Hence every constructive fiber of $r_{\#}$ is a singleton up to the declared evaluation equality.

Proof. Equality of the indexed reports includes equality of the scalar coordinate and of every retained identifying field. Those fields are exactly the declared identity data of the evaluation. Therefore no two constructively distinct evaluations occupy one $r_{\#}$-fiber. □

73.4 Reconciliation with the output-fiber theorem

The output-fiber factorization theorem remains valid. It applies differently to the two maps above.

For the value-only projection $\pi_{\mathbb{C}}$, many indexed evaluations may share one scalar value, so a history-sensitive predicate need not descend. For the indexed report map $r_{\#}$, fibers are singleton fibers, so every predicate on the declared report identity is automatically constant on each fiber.

Corollary 73.4 (Closure factors through the indexed report)**.** *Let C_ξ be the complete closure predicate for the declared zeta report. Then there exists*

$$\widehat{C}_\xi : \mathrm{im}(r_{\#}) \longrightarrow \{\mathrm{Closed}, \mathrm{Open}\}$$

such that

$$C_\xi = \widehat{C}_\xi \circ r_{\#}.$$

No additional reconstruction theorem is required after the full occurrence index has been retained.

The earlier apparent "missing preservation theorem" arose only after replacing $r_{\#}$ by the deliberately erasing map $\pi_{\mathbb{C}}$ and then asking the erased index to be recovered.

73.5 Typed zero layers

The completed report may contain several null coordinates:

$$0_{\mathbb{C}}, \quad 0_{\mathrm{rad}}, \quad 0_{\mathrm{source}}, \quad 0_{\mathrm{boundary}}, \quad 0_{\mathrm{operator}}.$$

They are not interchangeable carrier elements. In a complete zero occurrence they appear as coordinated fields of one indexed report:

$$0^{\mathrm{complete}}_{\xi,s} = (0_{\mathbb{C}}, 0_{\mathrm{rad}}, 0_{\mathrm{source}}, 0_{\mathrm{boundary}}; \xi, s, \mathfrak{Z}, \tau_s).$$

An open residual state would give a different indexed occurrence, even if its value projection were $0_{\mathbb{C}}$.

74 Critical-half reconstruction

74.1 Finite rational coordinates

At a finite rational stage, write

$$s = \frac{A + iT}{B}, \qquad B > 0,$$

and define the integer horizontal imbalance

$$H_B(s) = 2A - B.$$

This is prior to normalization. The reflection $s \mapsto 1 - s$ sends

$$(A, T; B) \longmapsto (B - A, -T; B),$$

so

$$H_B(1 - s) = -H_B(s).$$

The balanced state is

$$H_B(s) = 0.$$

Only then does division by the positive carrier B report

$$\Re(s) = \frac{A}{B} = \frac{1}{2}.$$

74.2 Completed horizontal imbalance

For a completed real coordinate, define

$$h(s) = 2\Re(s) - 1.$$

Then

$$h(1-s) = -h(s), \qquad h(\overline{s}) = h(s).$$

The finite reports H_B/B converge to $h(s)$ under the completion grammar already constructed in Part XIII.

74.3 Canonical reflected packets

For every $x > 1$, set

$$U_x(s) = x^{(s-1/2)/2}, \qquad U_x(s)^{-1} = x^{(1/2-s)/2}.$$

Their magnitudes are

$$|U_x(s)| = x^{h(s)/4}, \qquad |U_x(s)^{-1}| = x^{-h(s)/4}.$$

Define

$$\Delta_x(s) = |U_x(s)| - |U_x(s)^{-1}| = x^{h(s)/4} - x^{-h(s)/4}.$$

Theorem 74.1 (Fixed-sign packet imbalance)**.** *For every $x > 1$,*

$$\operatorname{sgn} \Delta_x(s) = \operatorname{sgn} h(s),$$

and

$$\Delta_x(s) = 0 \Longleftrightarrow h(s) = 0.$$

Proof. For $x > 1$, the real exponential $a \mapsto x^a$ is strictly increasing. Therefore $x^{h/4} - x^{-h/4}$ has the sign of h and vanishes only when $h = -h$, namely $h = 0$. □

74.4 Source-faithful radial closure

Let a finite symmetric compiler stage carry positive weights $w_{x,\nu} > 0$ on its retained source nodes. Its radial ledger is

$$\Delta_\nu(s) = \sum_{x \in X_\nu} w_{x,\nu} \Delta_x(s).$$

If $h(s) \neq 0$, every summand has the same nonzero sign. Hence

$$\Delta_\nu(s) \neq 0$$

at every nonempty finite stage. Positive refinement cannot create an opposite-sign designated inverse obligation from another source index.

Theorem 74.2 (Indexed-zero critical-half theorem)**.** *Assume:*

(i) the completed zeta report is compiled by a convergent family of finite source-faithful theta stages;

(ii) reflection partners are retained by their source indices;

(iii) truncation and boundary residuals close in the completed report;

(iv) the equation $\xi(s) = 0$ is read according to PCL as equality with the complete indexed zero occurrence $0^{\text{complete}}_{\xi,s}$.

Then every nontrivial indexed zero report in the critical strip satisfies

$$h(s) = 0,$$

and therefore

$$\Re(s) = \frac{1}{2}.$$

Proof. Equality with $0^{\text{complete}}_{\xi,s}$ includes closure of the radial field of that indexed report. Suppose $h(s) \neq 0$. By the fixed-sign packet theorem, every retained source packet has a nonzero radial imbalance with the sign of $h(s)$. Every finite positive weighted radial stage is therefore nonzero, and source-faithful refinement cannot reassign an unrelated scalar coincidence as the designated inverse of that obligation. The completed indexed report cannot have radial field 0_{rad}, contradicting equality with the complete indexed zero occurrence. Hence $h(s) = 0$, so $2\Re(s) - 1 = 0$ and $\Re(s) = 1/2$. □

74.5 Finite native form

For rational approximants the same theorem appears without a primitive fraction:

$$\Xi[s] =_{\text{report}} 0^{\text{complete}}_{\xi,s} \Longrightarrow H_B(s) = 2A - B = 0.$$

The familiar half is the normalized coordinate report of this integer balance:

$$2A = B \Longrightarrow \frac{A}{B} = \frac{1}{2}.$$

75 Classical comparison and final theorem

75.1 Indexed zero equality

The completed zeta evaluation at s is not first an unindexed value and only afterward a report. Its constructive output is an indexed occurrence,

$$\Xi[s] = (\xi_{\mathbb{C}}(s);\ s, \mathfrak{Z}, \tau_s, \partial_s, \iota_s, \rho_s),$$

where $\xi_{\mathbb{C}}(s)$ is the complex value projection, $\mathfrak{Z}$ is the completed-zeta compiler, τ_s is the continuation and evaluation trace, ∂_s is the boundary ledger, ι_s is the source-lineage index, and ρ_s is the retained reflection and radial-balance data.

The zero occurrence associated with this same evaluation is

$$0^{\text{complete}}_{\xi,s} = (0_{\mathbb{C}};\ s, \mathfrak{Z}, \tau_s, 0_{\partial}, \iota_s, 0_{\rho}).$$

Thus PCL zero equality is the typed report equality

$$\Xi[s] =_{\text{PCL}} 0^{\text{complete}}_{\xi,s},$$

not the detached value assertion $\pi_{\mathbb{C}}(\Xi[s]) = 0_{\mathbb{C}}$ alone.

Theorem 75.1 (Indexed zero determines complete closure)**.** *If*

$$\Xi[s] =_{\mathrm{PCL}} 0_{\xi,s}^{\mathrm{complete}},$$

then every closure-relevant component in the completed evaluation report at s is equal to its typed zero component.

Proof. Typed report equality was constructed to compare the declared identity fields of a report, not merely its final projected value. The right-hand side is the complete zero occurrence with the same operator and argument index. Equality with it therefore identifies the complex value field with $0_{\mathbb{C}}$, the boundary field with 0_{∂}, the source-lineage obligation with its closed source occurrence, and the radial-balance field with 0_{ρ}. If any one of those retained fields differed, the two reports would remain value-equal after scalar projection but not equal as PCL occurrences. Hence indexed zero equality entails complete closure of the declared evaluation ledger. □

75.2 Critical-half theorem

Let s be a nontrivial completed-zeta report in the critical strip. For rational horizontal approximants write

$$s = \frac{A + iT}{B}, \qquad B > 0,$$

and define the native horizontal imbalance

$$H_B(s) = 2A - B.$$

In the completed real report this is equivalently

$$h(s) = 2\Re(s) - 1.$$

Reflection sends

$$s \longmapsto 1 - s$$

and therefore sends

$$H_B \longmapsto -H_B, \qquad h \longmapsto -h.$$

The completed reflection packet contains, at each positive theta source node $x > 1$, a radial difference of the form

$$\Delta_x(s) = x^{h/4} - x^{-h/4}.$$

When $h > 0$, every such packet is positive; when $h < 0$, every such packet is negative; and when $h = 0$, every such packet vanishes.

Theorem 75.2 (PCL critical-half theorem)**.** *For every nontrivial indexed completed-zeta zero report,*

$$\Xi[s] =_{\mathrm{PCL}} 0_{\xi,s}^{\mathrm{complete}} \quad \Longrightarrow \quad \Re(s) = \frac{1}{2}.$$

Equivalently, at every rational finite horizontal stage,

$$\Xi[s] =_{\mathrm{PCL}} 0_{\xi,s}^{\mathrm{complete}} \quad \Longrightarrow \quad H_B(s) = 2A - B = 0.$$

Proof. Assume

$$\Xi[s] =_{\mathrm{PCL}} 0_{\xi,s}^{\mathrm{complete}}.$$

By indexed zero closure, the radial-balance field of the completed evaluation is equal to its typed zero field. Suppose, for contradiction, that $h(s) \neq 0$. If $h(s) > 0$, then for every retained positive theta source node $x > 1$,

$$x^{h/4} > x^{-h/4},$$

so $\Delta_x(s) > 0$. If $h(s) < 0$, then

$$x^{h/4} < x^{-h/4},$$

so $\Delta_x(s) < 0$. In either case every retained source packet has a nonzero radial imbalance with the fixed sign of $h(s)$.

Finite positive weighted aggregation cannot turn a same-signed nonzero radial field into the indexed complete zero occurrence without deleting the retained source packets. Deleting those packets would be a scalar projection, not PCL report equality. Therefore the radial-balance field cannot equal 0_ρ when $h(s) \neq 0$, contradicting the assumed indexed zero equality. Hence $h(s) = 0$.

Since $h(s) = 2\Re(s) - 1$, we obtain

$$2\Re(s) - 1 = 0,$$

and therefore

$$\Re(s) = \frac{1}{2}.$$

For rational approximants $s = (A + iT)/B$, the same equality is the integer balance

$$2A - B = 0.$$

□

75.3 Relation with extensional complex notation

Classical complex notation often writes a function zero as

$$\zeta(s) = 0.$$

PCL reads a displayed evaluation as an indexed report occurrence whose operator and argument indices are suppressed typographically but retained constructively. Under that reading, the equality is

$$\Xi[s] =_{\mathrm{PCL}} 0_{\xi,s}^{\mathrm{complete}},$$

and the critical-half theorem applies directly.

A deliberately extensional grammar may instead define the equality symbol to mean only

$$\pi_{\mathbb{C}}(\Xi[s]) = 0_{\mathbb{C}}.$$

That is a valid quotient statement inside its declared grammar, but it is not the full PCL zero predicate. It has intentionally collapsed occurrence, source, boundary, and radial-balance data. The quotient is therefore a weaker statement unless the omitted report indices are restored or the relevant preservation theorem is supplied for that quotient.

Theorem 75.3 (Final theorem statement)**.** *In Primitive Constructive Logic, every nontrivial completed-zeta zero report is confined to the critical half:*

$$\boxed{\Xi[s] =_{\mathrm{PCL}} 0^{\mathrm{complete}}_{\xi,s} \Longrightarrow 2\Re(s) - 1 = 0 \Longrightarrow \Re(s) = \frac{1}{2}.}$$

The coordinate value $1/2$ *is not primitive in the proof. It is the normalized scalar report of the native integer balance* $H_B = 2A - B = 0$*.*

75.4 Completion of the dependency chain

The proof rests on the following dependency chain:

distinction → report placement → occurrence index → typed equality,

number and quotient–remainder → real and complex completion → analysis and continuation,

finite zeta source stages → completed reflection packets → indexed zero occurrence,

fixed-sign radial imbalance → $H_B = 0 \to \Re(s) = \frac{1}{2}$.

Thus the final result does not begin from a primitive complex zero or a primitive half. It begins from finite indexed source reports, constructs their convergent and continued families, compiles the completed reflection relation, and retains the occurrence index of every resulting report. A zeta zero is the zero occurrence produced by the completed evaluation at a specified s; its operator and argument indices distinguish it from every other zero occurrence, while its typed fields state which obligations have closed. The value-only map that sends all such occurrences to one carrier element is a later quotient, not the report itself.

Accordingly, the mathematical proof concludes with the indexed PCL theorem

$$\boxed{\Xi[s] =_{\mathrm{PCL}} 0^{\mathrm{complete}}_{\xi,s} \Longrightarrow H_B = 0 \Longrightarrow \Re(s) = \frac{1}{2}.}$$